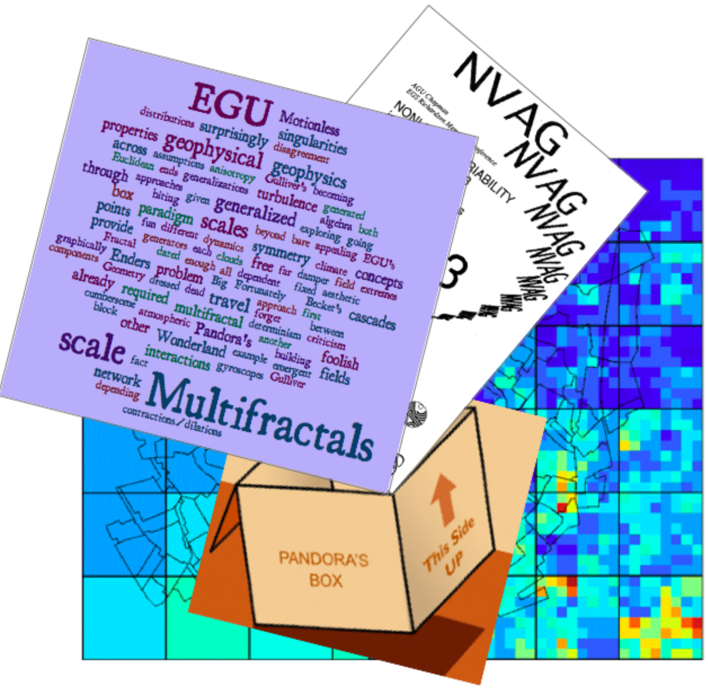
Beyond scalar modelling of multifractal precipitation and wind fields



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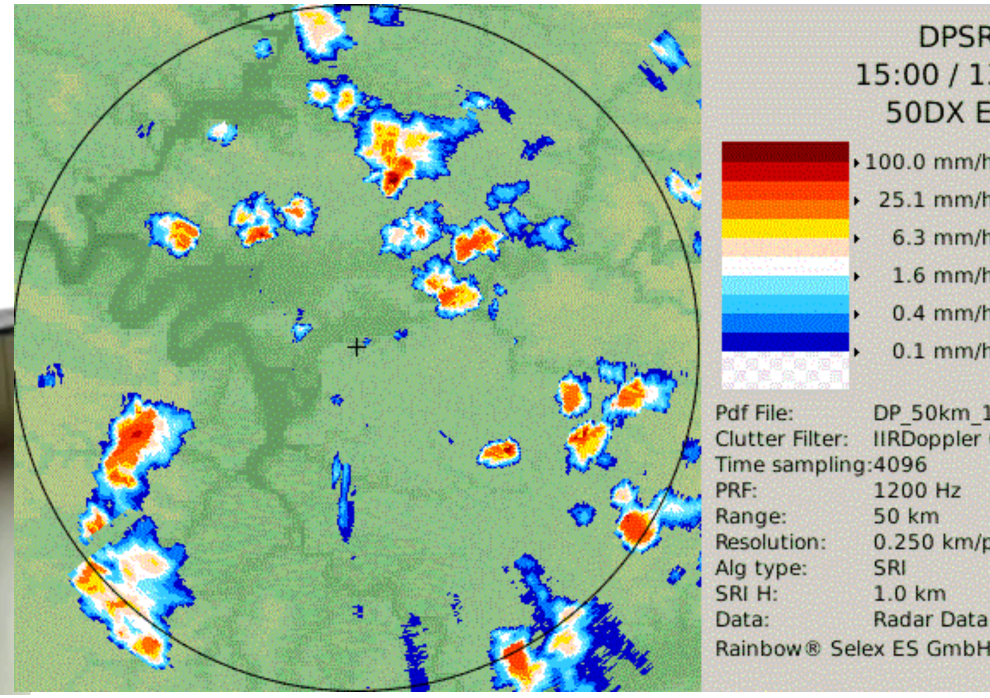
Ecole des Ponts ParisTech

Ra2DW Kickoff
5 February 2024

Millenium problem of turbulence !

exploratorium®



Art piece 'Windswept' (Ch. Sowers, 2012): 612 freely rotating wind direction indicators to help a large public to understand the complexity of environment near the Earth surface

Polarimetric radar observations of heavy rainfalls over Paris region during 2016 spring (250 m resolution):

- **heaviest rain cells** are much smaller than **moderate ones**
- **complex dynamics** of their aggregation into a large front

Scale symmetry and equations

Whereas elementary mathematical properties of Navier-Stokes solutions are still unknown (existence, uniqueness):

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{grad}(\underline{u}) = \underline{f} - \frac{1}{\rho}\underline{grad}(p) + \nu \Delta \underline{u}$$

one can point out a **scale symmetry** (*):

$$T_\lambda: \underline{x} \mapsto \underline{x}/\lambda$$

$$\tilde{T}_\lambda: \begin{array}{r} t \mapsto t/\lambda^{1-\gamma}; \underline{u} \mapsto \underline{u}/\lambda^{\gamma}; \\ \underline{f} \mapsto \underline{f}/\lambda^{2\gamma-1}; \nu \mapsto \nu/\lambda^{1+\gamma}; \rho \mapsto \rho/\lambda^{\gamma'} \end{array}$$

Kolmogorov's scaling (K41) obtained with:

$$\varepsilon(\ell) \approx \frac{\delta u^3}{\ell} \approx \bar{\varepsilon} \Rightarrow \gamma_K = 1/3$$

General case: multiple singularities γ's:

$$\Pr(\gamma' > \gamma) \approx \lambda^{-c(\gamma)}$$

(*) from to self-similarity (Sedov, 1961), symmetry (Parisi +Frisch, 1985), to generalised Galilean invariance (S+al, 2010)

Beyond scalar rain rate?

École des Ponts ParisTech

HYDROLOGY METEOROLOGY and COMPLEXITY

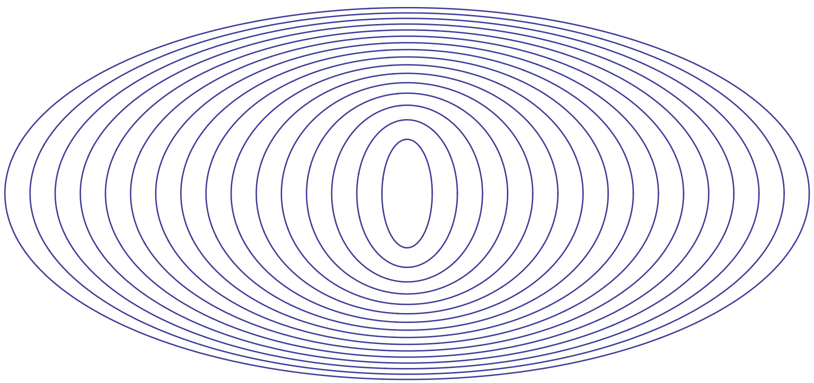
– multifractals clarified the *singular* behaviour of the rain rate: a *mathematical measure*, not a *function,* rather a *flux accros scales*

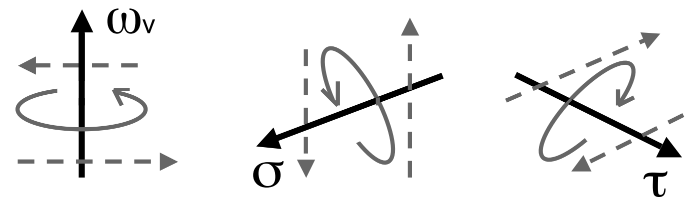
- new step: not only along the vertical
- require multi-component/multivariate cascades
  - *vector valued/multivariate* multifractals
  - more generally: manifold valued multifractals
- generic solutions combine
  - scaling anisotropy
  - robust statistics (Lévy stable processes)
  - anti-commutation structural properties (Clifford algebras)

Scaling anisotropy: $2+H_z$-dimensional vorticity equation

$(0<H_z<1)$

Scaling stratified /convective atmosphere:





$$D\vec{\sigma}/Dt = (\vec{\sigma}\cdot\vec{\nabla}_h)\vec{u}_h$$

$$D\vec{\tau}/Dt = (\vec{\tau}\cdot\vec{\nabla}_h + \boxed{\vec{\omega}_v\cdot\vec{\nabla}_v)\vec{u}_h}$$

$$D\vec{\omega}_v/Dt = (\vec{\tau}\cdot\vec{\nabla}_h + \vec{\omega}_v\cdot\vec{\nabla}_v)\vec{u}_v$$

Strong interactions between *local generalized* scales,

= *strongly non local* (Euclidean) scales !

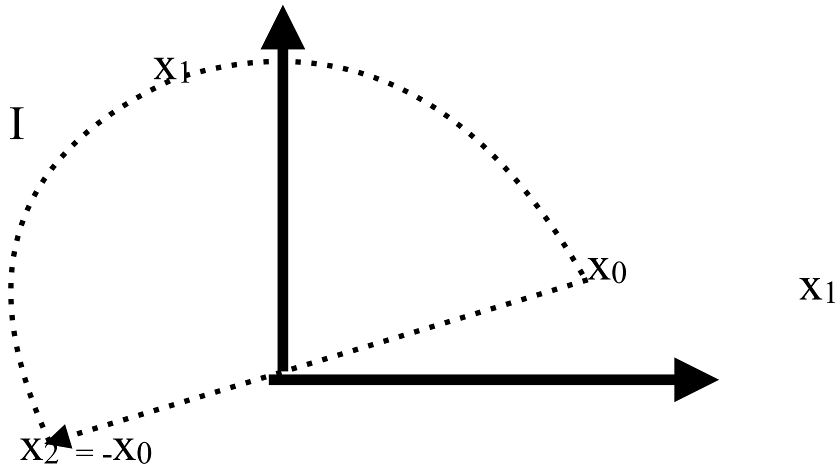
- a difficulty for direct numerical simulations ?
- easy for stochastic simulations !

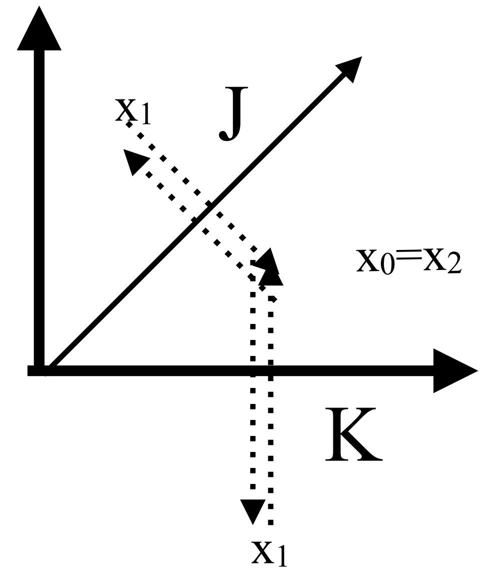
S&al, APC, 2012

Symmetries and unity roots

I=rot ($\pi/2$)

J, K =mirror symmetries





$I^2=-1$

$K^2=J^2=1$

Symmetries and unity roots

$I = \text{rot}\ (\pi/2)$

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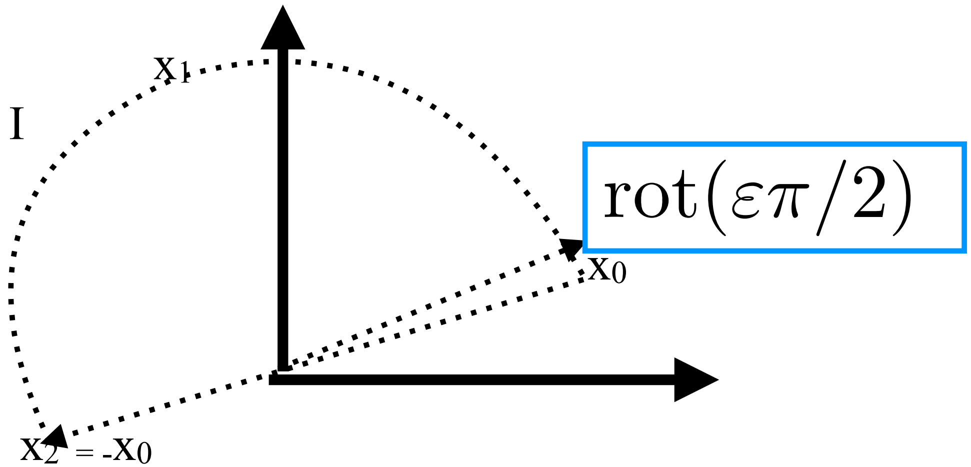

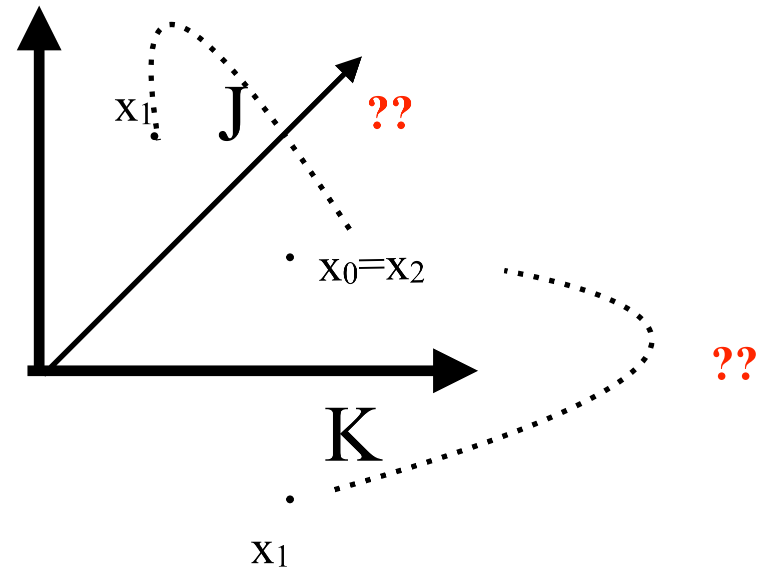

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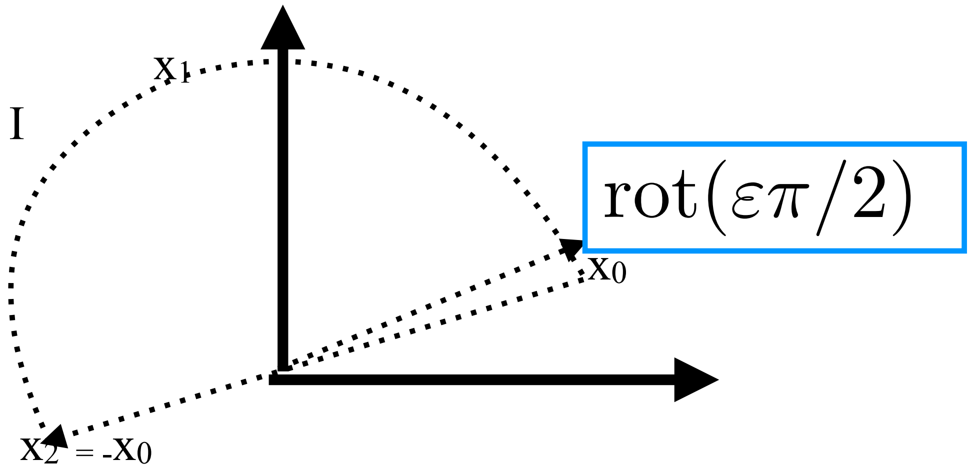

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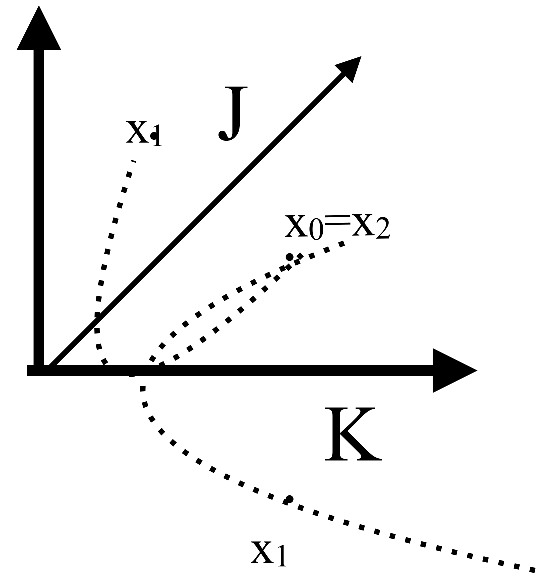

$I^2 = -1$

$K^2 = J^2 = 1$

Spherical geometry —> Hyperbolic geometry

Combining symmetries

2D linear Lie algebra H'= *l*(2, R):

$$G = d\mathbf{1} + e\mathbf{I} + f\mathbf{J} + c\mathbf{K};$$

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$2I = [J, K], \quad 2J = [I, K], \quad 2K = [J, I]$$

anti-commutators:

$$\{I, J\} = \{J, K\} = \{K, I\} = 0$$

$$I^2 = -J^2 = -K^2 = IJK = -1$$

(pseudo or split quaternions)

"quaternion equation" (Hamilton, 16/10/1843)

$$I_2^2 = J_2^2 = K_2^2 = I_2 J_2 K_2 = -1$$

Combining symmetries

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anti-commutators:

$$\{I, J\} = \{J, K\} = \{K, I\} = 0$$

$$I^2 = -J^2 = -K^2 = IJK = -1$$

$$I_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; J_2 = \begin{bmatrix} 0 & -K \\ K & 0 \end{bmatrix}; K_2 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

“quaternion equation” (Hamilton, 16/10/1843)

$$I_2^2 = J_2^2 = K_2^2 = I_2 J_2 K_2 = -1$$

Algebra of cascade generators

- Clifford algebra, dimension = 2^n
  - real numbers *R* (n=0), complex numbers *C* (n=1), quaternions *H* (n=2) other hyper-complex numbers, external algebras and many more!
- $Cl_{p,q}$: generated by operators $\{e^i\}$ that anti-commute and square to plus or minus the identity:

$$e^i e^j = -e^j e^i \ (i \neq j) \qquad (e^i)^2 = \pm 1$$

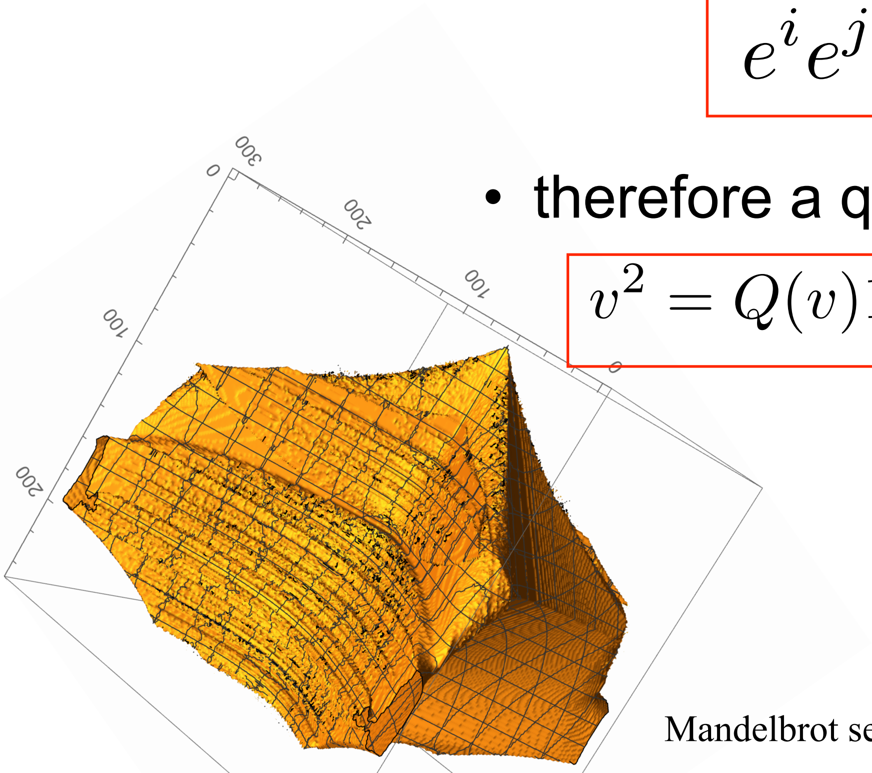
- therefore a quadratic form *Q* of signature (p,q, $p+q=n$):

$$v^2 = Q(v)1 \quad Q(v) = v_1^2 + v_2^2 .. + v_p^2 - v_{p+1}^2 - v_{p+2}^2 .. - v_{p+q}^2$$

ex.: $R = Cl_{0,0}$; $C = Cl_{0,1}$; $H = Cl_{0,2}$

H'= $I(2, \mathrm{R}) = Cl_{2,0} = Cl_{1,1}$

"pseudo-/split- quaternions"



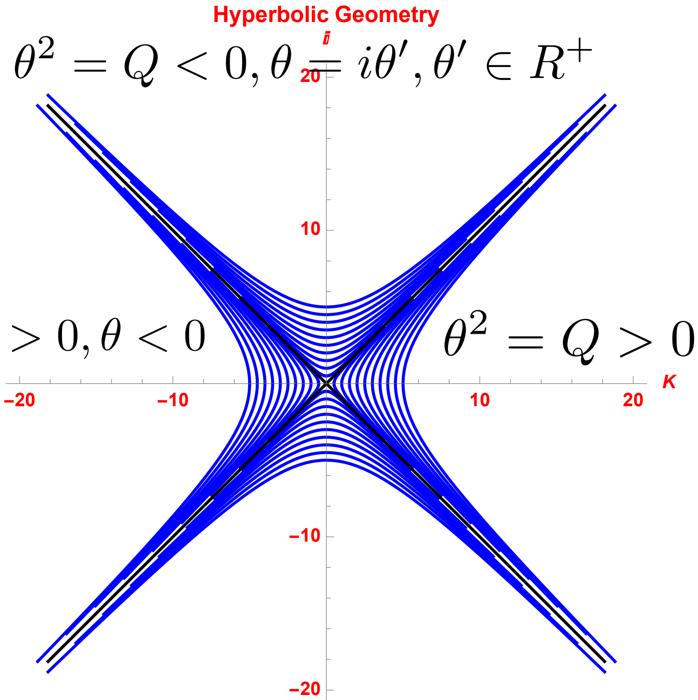
Mandelbrot set on pseudo-quaternions, S&T,2018

From algebra to group

{K, I}

$\theta^2 = Q < 0, \theta = i\theta', \theta' \in R^+$

$\theta^2 = Q > 0, \theta < 0$

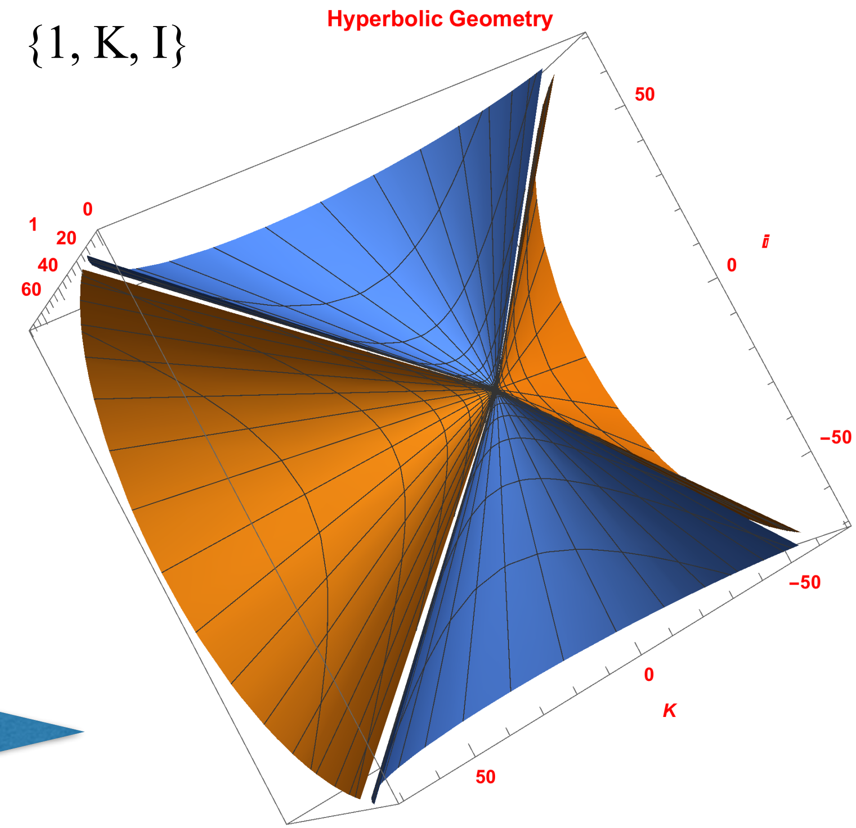


$\theta^2 = Q > 0, \theta > 0$

$\theta^2 = Q < 0, \theta = i\theta', \theta' \in R^-$

Exp →

{1, K, I}



Generalised Moivre-Euler formula: $(e^{u\theta})^{\alpha} = \cosh(\alpha\theta)1 + \sinh(\alpha\theta)u$

infinite number of u, $u^2 = \pm 1$!

Stochastic Clifford?

- Statistical universality: stable Lévy vectors

$$\forall n \in N, \exists a(n), b(n) \in R : \sum_{i=1}^{n} X_i =^d a(n)X + b(n)$$

$$\exists \alpha \in (0,2] : a(n) = n^{1/\alpha}; \alpha < 2, \forall s \gg 1 : \mathrm{P}([X| > s) \approx s^{-\alpha}$$ (hyperbolic/Pareto tail)

$\alpha = 2 :$ Gauss

A stable Levy *X* is attractive for any Y_i having same type of tail:

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} Y_i - b(n)}{a(n)} =^d X$$

– classical "quasi- scalar" case: only *b is a* vector like X_i and Y_i

– 'strong' vector case: *a* and α are matrices *(S. et al., 2001)*

Exponentiation of Lévy-Clifford algebra

- Existence ?
  - Q defines a bilinear form < . >

$$< X, Y > = \frac{1}{2}\left(Q(X+Y) - Q(X) - Q(Y)\right)$$

  - which defines a Laplace-Clifford transform,
  - hence a second characteristic function (cumulant generating function

$$\mathrm{E}\exp(< q, \Gamma_\lambda >) = Z_\lambda(q) = \exp(K_\lambda(q))$$

finite over set of cones $\mathscr{A}^{\downarrow}$

the opposite cones to that supporting the extremely assymetric Lévy stable component $\mathscr{A}^{\uparrow}$

Fractionnaly Integrated Flux model (FIF, vector version)

FIF assumes that both the renomalized propagator G_R and force f_R are known:

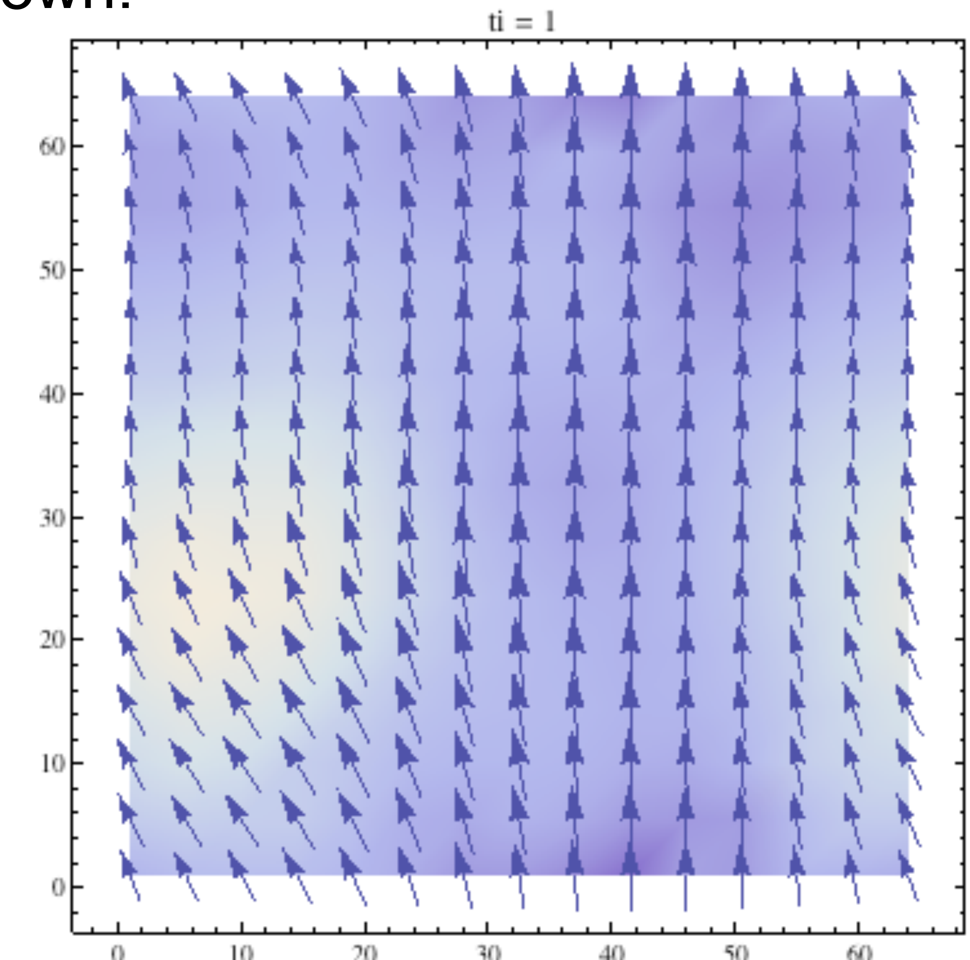
$$G_R^{-1} * u = f_R$$

where: $f_R = \varepsilon^a$

G_R^{-1} is a fractionnal differential operator

ε results from a continuous, vector, multiplicative cascade (Lie cascade)

Complex FIF simulation of a 2D cut of wind and its vorticity (color)



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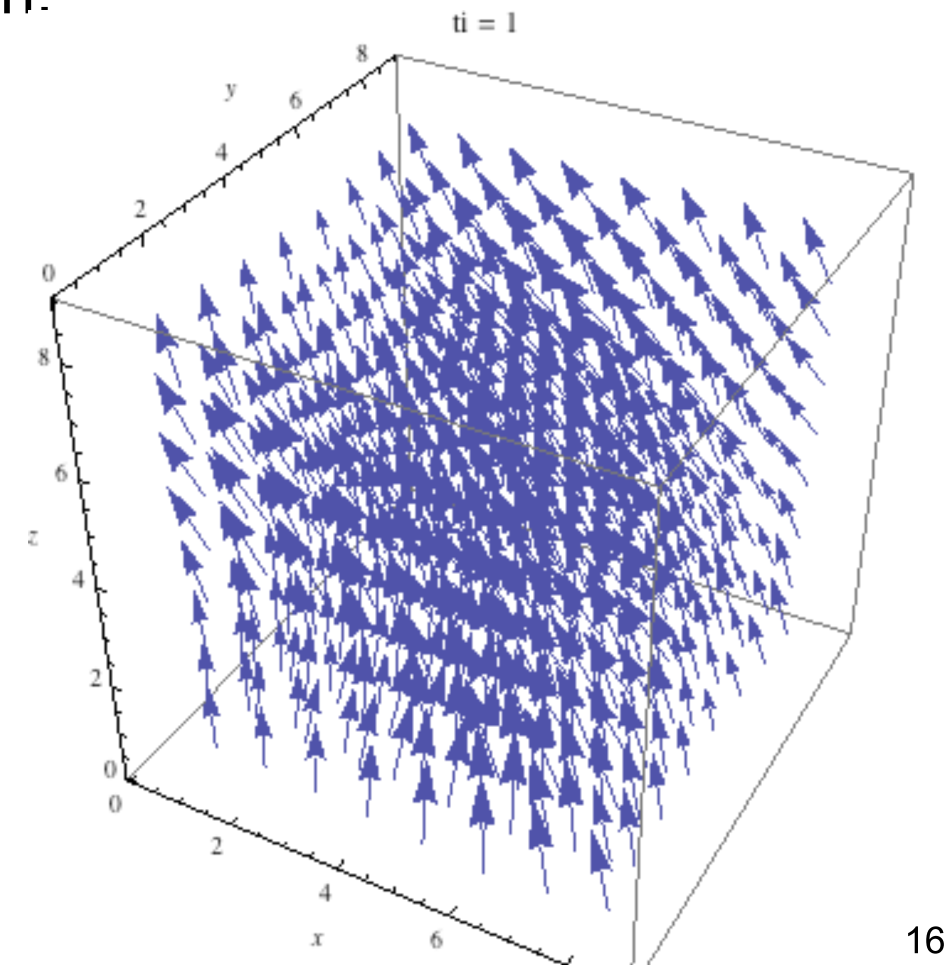
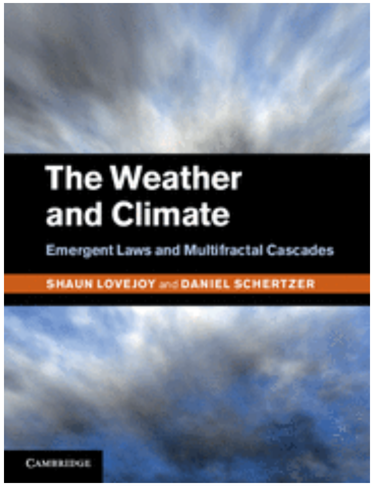
3D FIF wind simulation based on quaternions



Conclusions

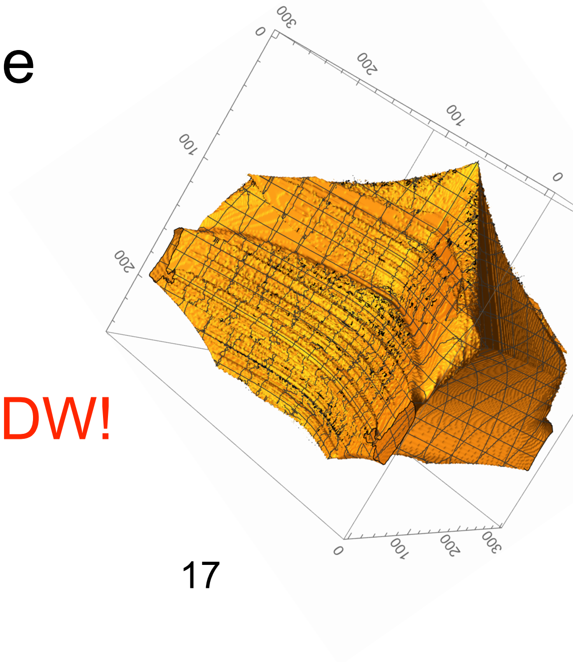
S&T, Earth& Space, 2020
Chaos 2015, S&al. ACP, 2012,
S&L, IJBC, 2011,
Fitton&al., JMI 2013



- Intermittency: a key issue in geophysics and a major breakthrough with multifractals in the 1980's:
  - infinite hierarchy of fractal supports of the field singularities
  - beyond commonalities significant differences of approaches and applications
- No longer limited to scalar valued fields
  - multifractal operators: exponentiation from a stochastic Lie algebra of generators onto its Lie group of transformations
  - ex. Clifford algebra $Cl_{p,q}$

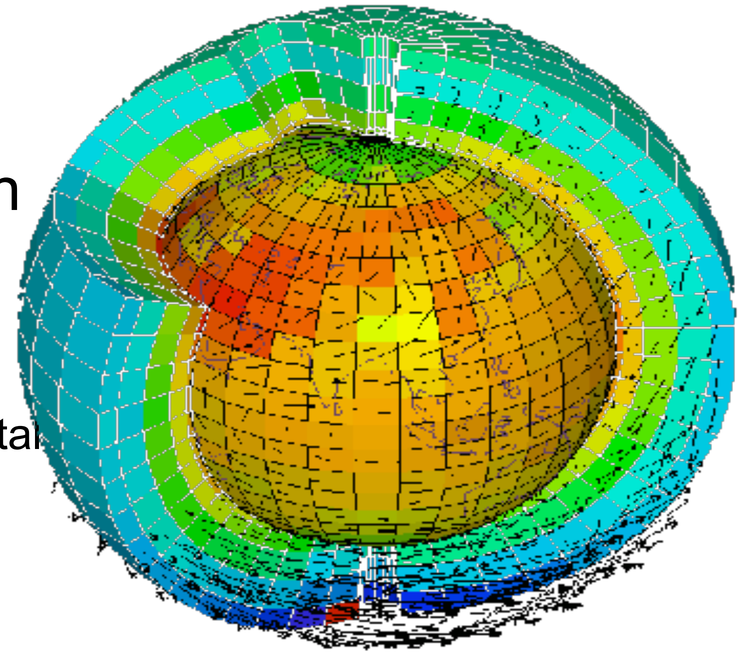
=> from field physics to singularity physics

multifractal multivariate rain rate to be tested within Ra2DW!

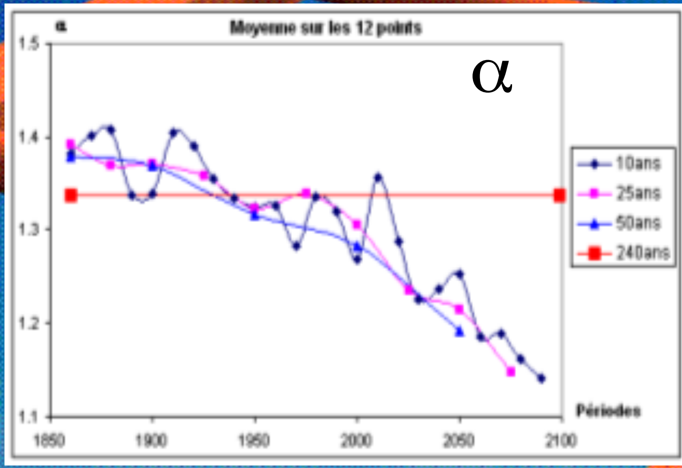


Beyond the curtain

- Multiscale complexity vs. computing brute force
  - (nonlinear) equations discretised on *N* voxels (≈cubes)
  - voxels of mm^3 to reach the ***viscous scale*** (≈1mm)
  - atmosphere ≈ 10 km high x $(10{,}000\ km)^2$ horizontal
  - => $N \approx 10^7(10^{10})^2 = 10^{27}$
  - much larger than $N_A = 10^{23}$!!
- How to be deterministic… over a small range of scales?

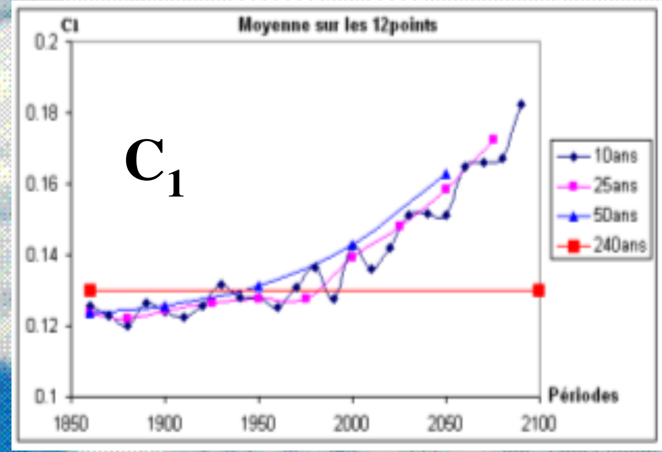


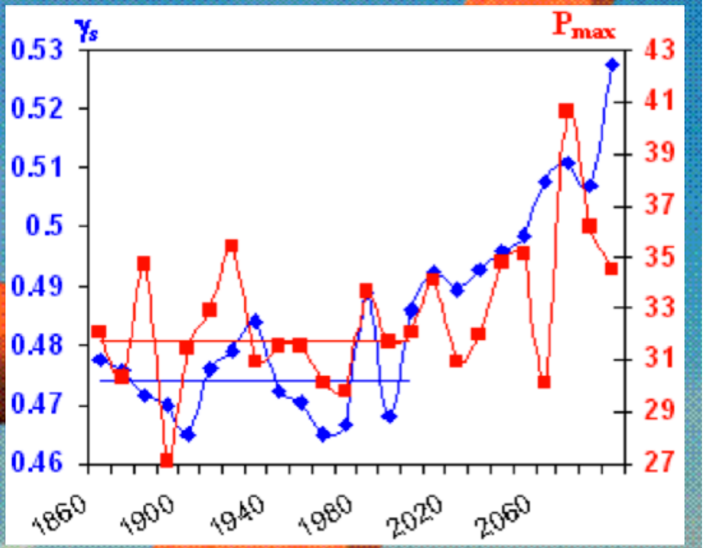
Multiscale analysis of the scenario A2 1860-2100 (CNRM-CM3)



- average intermittency C1 ↑
- intermittency variability α ↓,

=> difficulty to evaluate extremes of precipitations





=> refined analysis :

- time volution of the **Most Probable Singularity** γ_s (Hubert et al, 1993; Douglas & Barros, 2003):

- a scale invariant statistic, more stable than the maximal simulated precipitation P_{max}.

- Enable us to conclude: **extremes** ↑ (Royer et al., 2008),

-- seasonality can be taken into account (Royer et al., 2010)

From geometry to analytics

T_λ^* =pullback of T_λ for functions

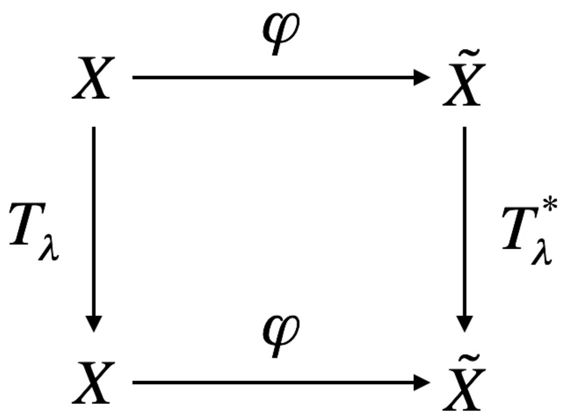


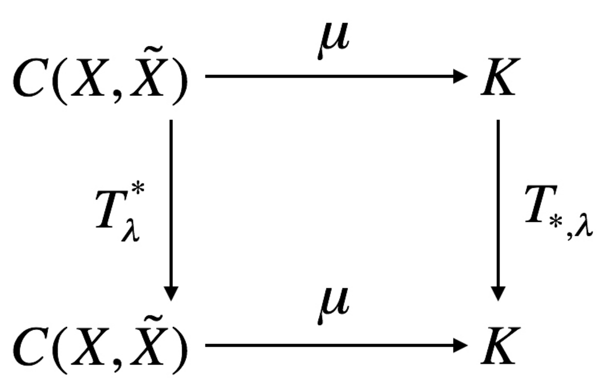
Figure 1: Commutative diagram illustrating how the analytical pullback transform T_λ^* is generated on the codomain $\tilde{X}$ of the field φ by the geometric transform T_λ on the domain X.

ex.: simple scaling (e.g. Lamperti, 1962)

$$T_\lambda x = x/\lambda,\ T_\lambda^* y = y/\lambda^H$$

$T_{*,\lambda}$ =push forward of T_λ

for measures or generalised functions



ex.: fractal measure of dimension D

$$T_\lambda x = x/\lambda,\ T_{*,\lambda}\mu = \mu/\lambda^D$$

Figure 3: Commutative diagram, similar to that of Fig. 1, illustrating how the analytical pullback transform T_λ^* generates in turn the push forward $T_{*,\lambda}$ for measures or generalized functions μ's.

From geometry to analytics

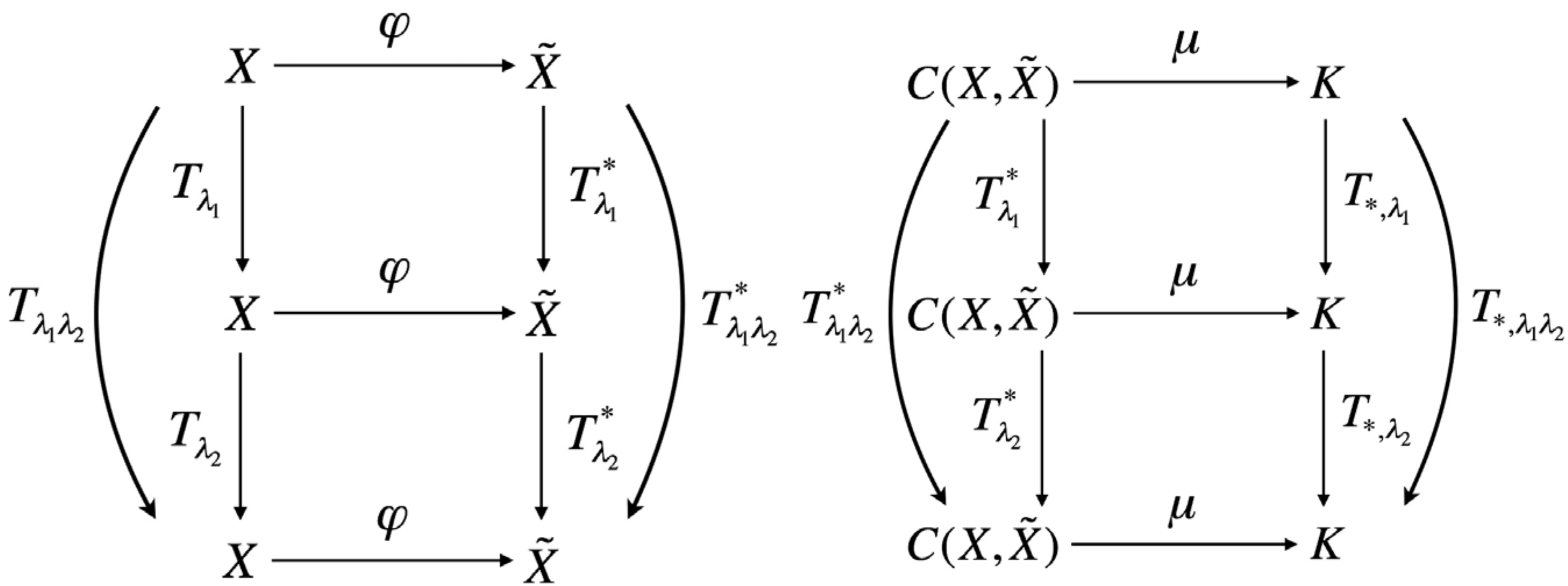
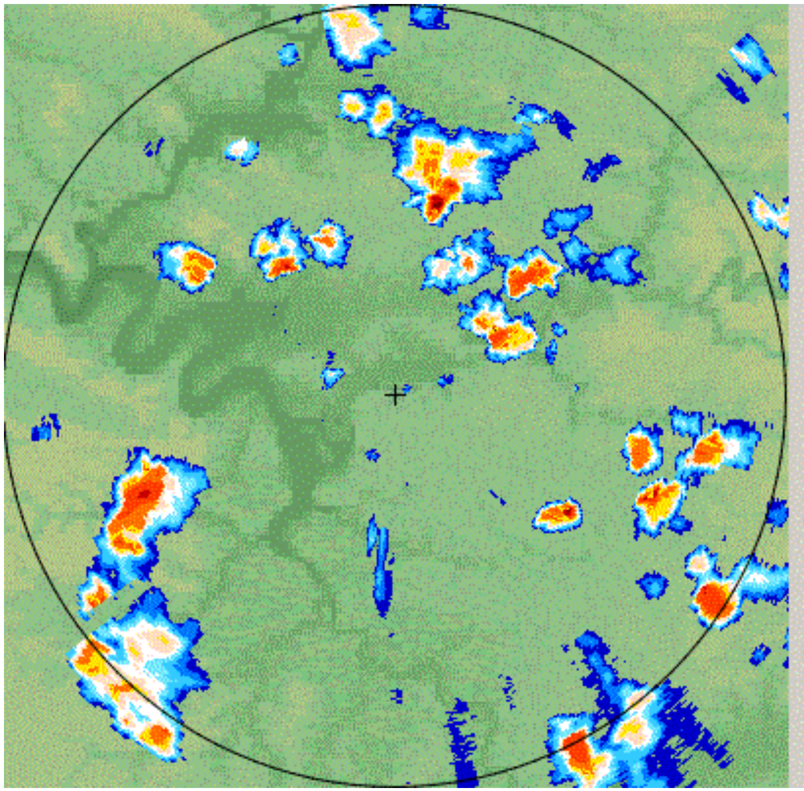


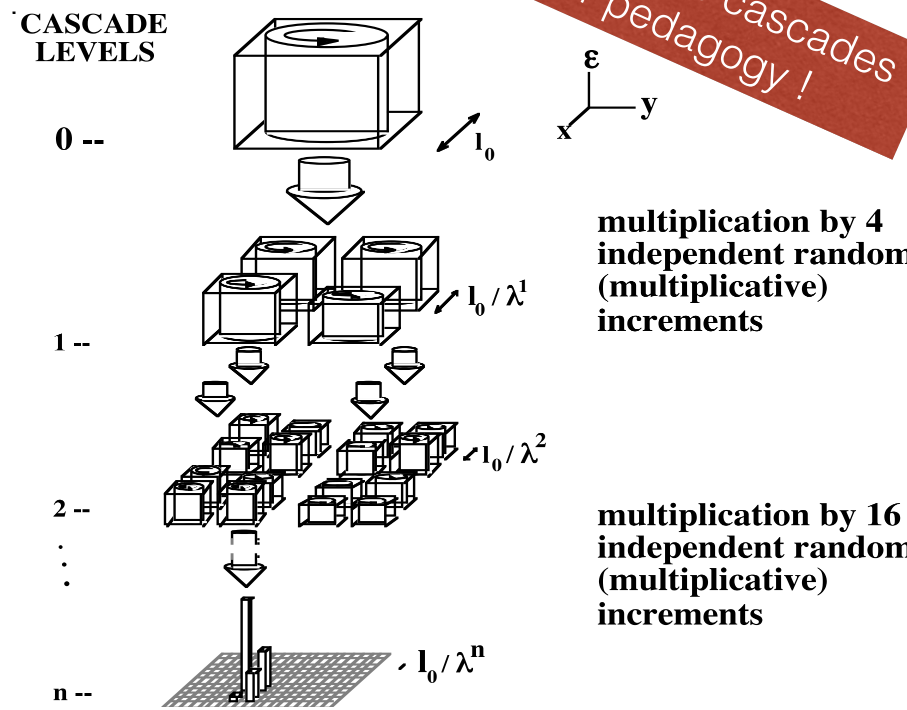
Figure 5: These diagrams show how the group property of T_λ propagates in a straightforward manner to the "pullback" transform T^*_λ (left) and then (by duality) to the "push forward" transform $T_{*,\lambda}$ (right).

Russian dolls… and multiplicative cascades

École des Pons

Discrete in scale cascades only for pedagogy !





Polarimetric radar observations of heavy rainfalls over Paris region during 2016 spring (250 m resolution):

- **heaviest rain cells** are much smaller than **moderate ones**
- true for their dimensions => **multifractal field**
- **complex dynamics** of their aggregation into a large front

3D Scaling Gyroscope Cascade

$$\left(\frac{d}{dt}+\nu k_n^{2}\right)\hat{u}_n^{i\,*}=i\{k_{n+1}[\left|\hat{u}_{n+1}^{2i-1}\right|^2-\left|\hat{u}_{n+1}^{2i}\right|^2]+(-1)^i k_n \hat{u}_n^{i\,*}\hat{u}_{n-1}^{a(i)}\}$$

$a(i)$ is an ancestor.

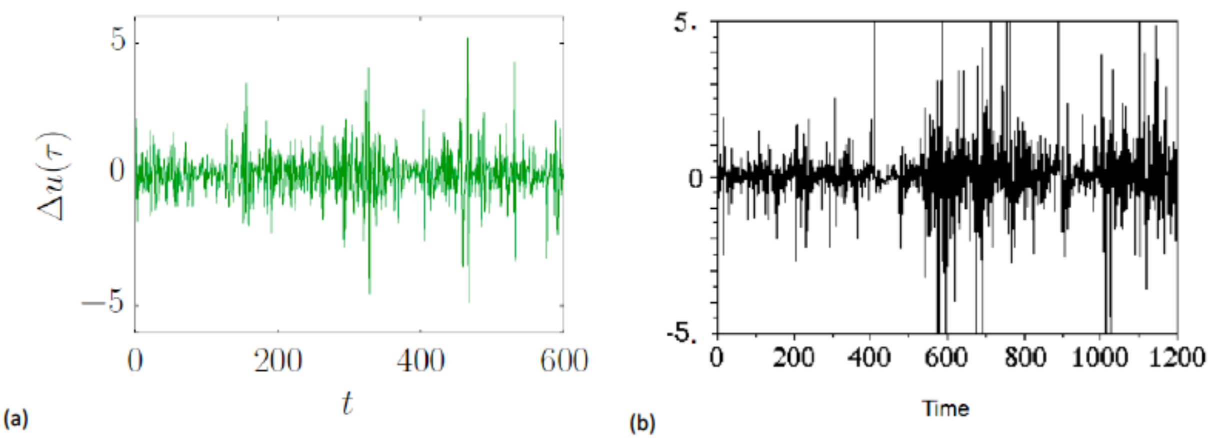
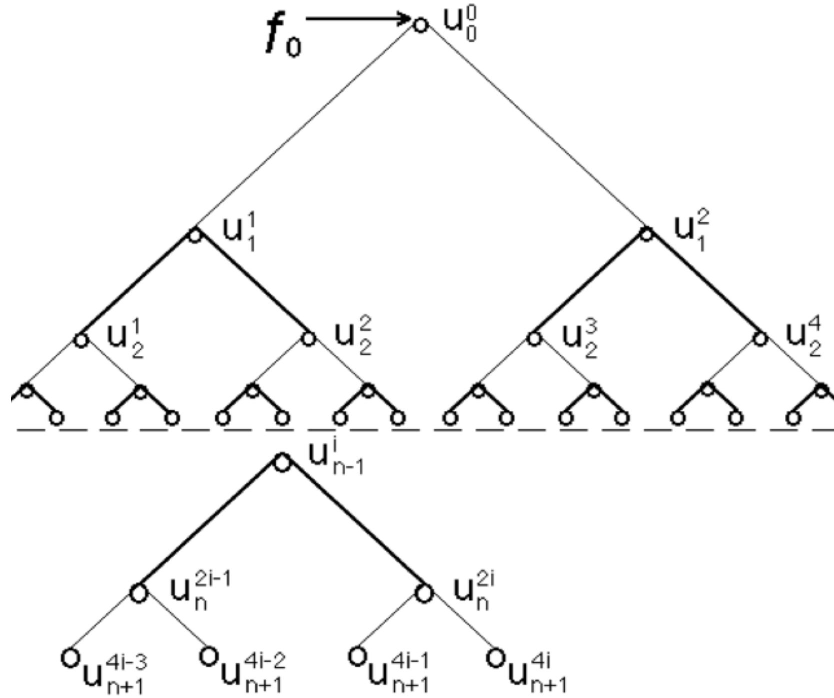


Figure 2: Comparison of fluctuations: (a) atmospheric turbulence at 100*m* (*Fitton*, 2013) and (b) SGC simulation for n=6 (*Chigirinskaya and Schertzer*, 1996), both display somehow similar strong intermittency.



Local flux of energy:

$$\varepsilon_n^i=-\sum_{r=0}^{n} k_{n-r+1}\left[\left|\hat{u}_{n-r+1}^{2a^r(i)-1}\right|^2-\left|\hat{u}_{n-r+1}^{2a^r(i)}\right|^2\right]\mathrm{Im}(\hat{u}_n^{a^r(i)})+(-1)^{a^r(i)+1}k_{n-r}\left|\hat{u}_n^{a^{r+1}(i)}\right|^2\mathrm{Im}(\hat{u}_{n-1}^{a^{r+1}(i)})$$

Mr Jourdain and Lie cascades

- Levi decomposition of any Lie algebra into its radical (*good guys*!) and a semi-simple subalgebra (*bad guys*!), e.g.:

$$l(2, R) = R\,1 \oplus_s sl(2, R)$$

What is trickier:

- large number of degrees of freedom (*dim*^{2})
- log divergence with the resolution
- universality:
  - Levy multivariates, unlike Gaussian mutivariates, are non parametric (*)
  - asymmetry of Levy noises to have convergent statistics, e.g.:

$$\forall n \in N, \forall X \geq 0 : \; \exp(X) \geq X^n / n!$$

(S&L, 95, T&S 96)

(*) limitation of anamorphosis transform and/or geostatistics

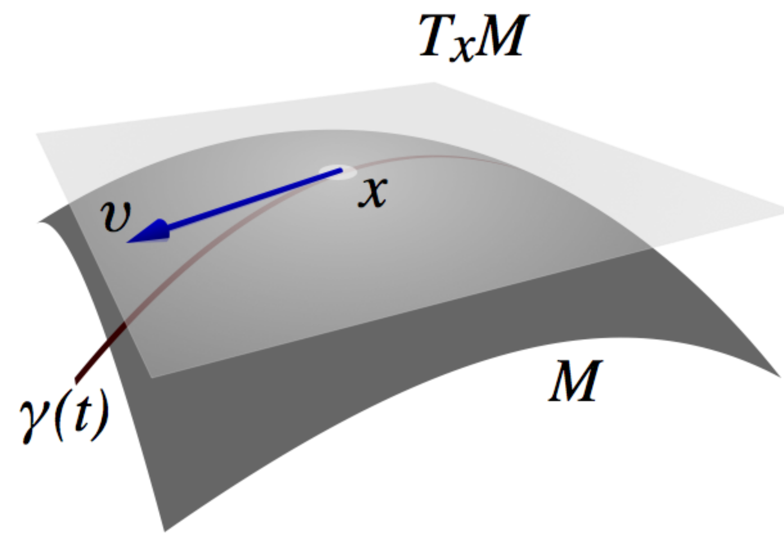
Mr Jourdain and Lie cascades

What is general and theoretically straightforward:

- $exp : Lie\ algebra \longmapsto Lie\ group$

scalar valued cascade: R^d --> R^+

- Lie group: smooth manifold
- Lie algebra: tangent space to the group at the identity
- therefore a vector space with a skew product that satisfy the Jacobi identity:



$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

exemple: commutator of matrices

$$[X,Y] = XY - YX \qquad [X,Y] = 0 \Rightarrow \exp(X+Y) = \exp(X)\exp(Y)$$

Clifford algebra

- An important family of Lie algebras of operators:
  - their dimension: 2^n
  - generalizes real numbers *R* (n=0), complex numbers *C* (n=1), quaternions *H* (n=2) and other hyper-complex numbers, external algebras and more!
- $Cl_{p,q}$ has a basis $\{e^i\}$ whose vectors anti-commute and square to plus or minus the identity:

$$e^i e^j = -e^j e^i \ (i \neq j) \qquad (e^i)^2 = \pm 1$$

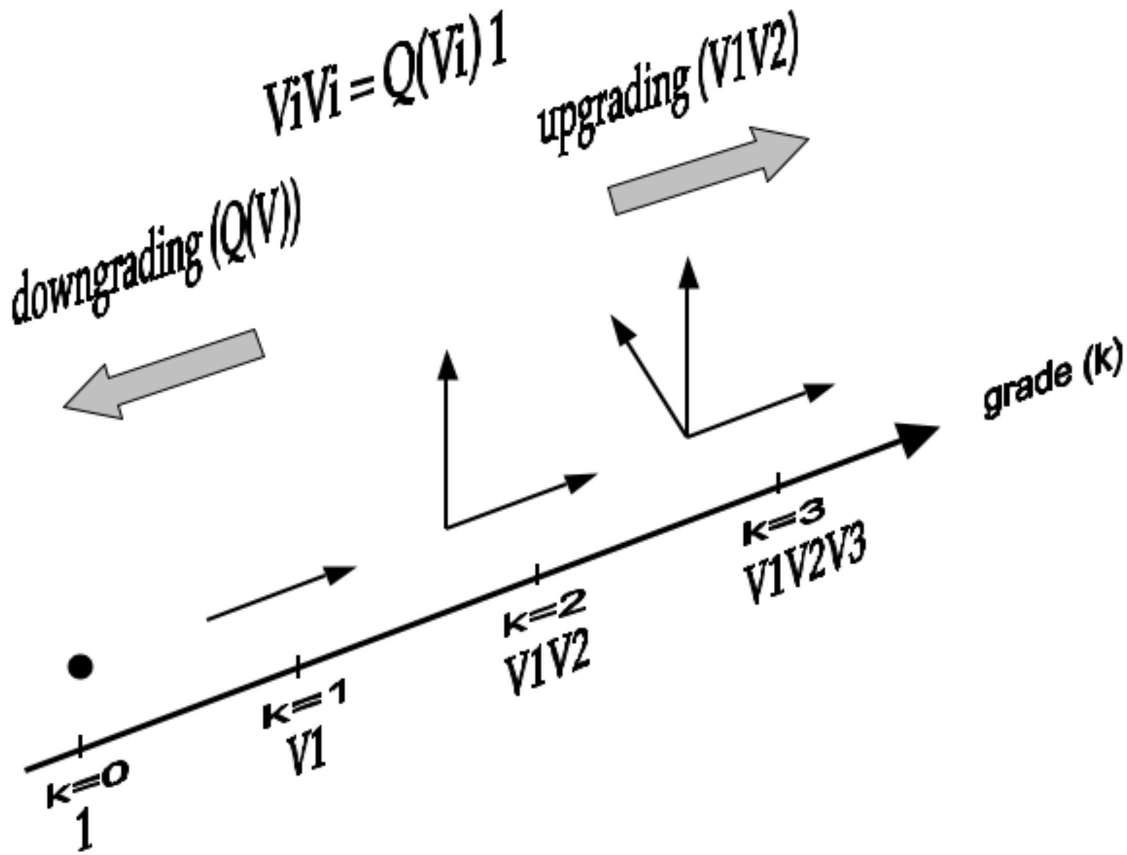
- it is generated by a n-dimensional vectorial space V={v} of operators and a quadratic form Q, of signature (p,q, $p+q=n$), which can be put into the canonical form:

$$v^2 = Q(v)1 \quad Q(v) = v_1^2 + v_2^2 .. + v_p^2 - v_{p+1}^2 - v_{p+2}^2 .. - v_{p+q}^2$$

ex.: $R = Cl_{0,0}$; $C = Cl_{0,1}$; $H = Cl_{0,2}$

H'= $l(2, R) = Cl_{2,0} = Cl_{1,1}$ "pseudo-/split- quaternions"

Clifford algebra



Clifford algebra are

- graded algebra (see figure)
- double algebra:
  - 2 multiplications
- super algebra (!):

$$Cl(V,Q) = Cl^0(V,Q) \oplus Cl^1(V,Q)$$

for real algebra:

$$Cl^0_{p,q}(R) \cong Cl_{p,q-1}(R) \text{ for } q > 0$$

$$Cl^0_{p,q}(R) \cong Cl_{q,p-1}(R) \text{ for } p > 0$$

==> $R \subset C \subset H \subset O \quad \ldots$