# Discrete and continuous multifractal cascade models: historical roots and application to turbulence

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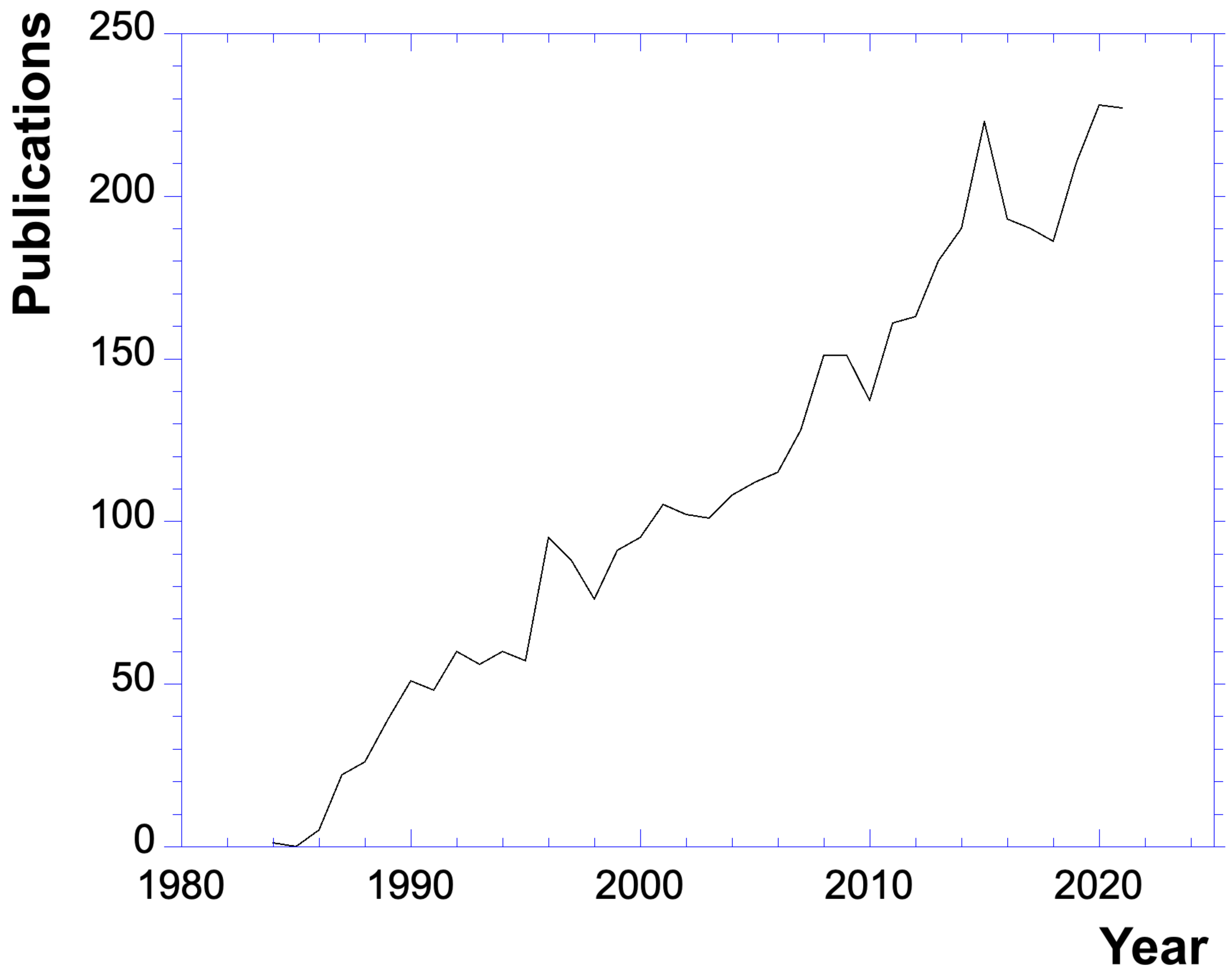
EGU Division Nonlinear Processes in Geosciences

EGU NP Division Campfire

Scaling and multifractals : from historical perspectives to recent developments




began in 1985 and regular increase since then

total = 4246

**Annual number of articles with « multifractal » in the title**

source: Web of Science

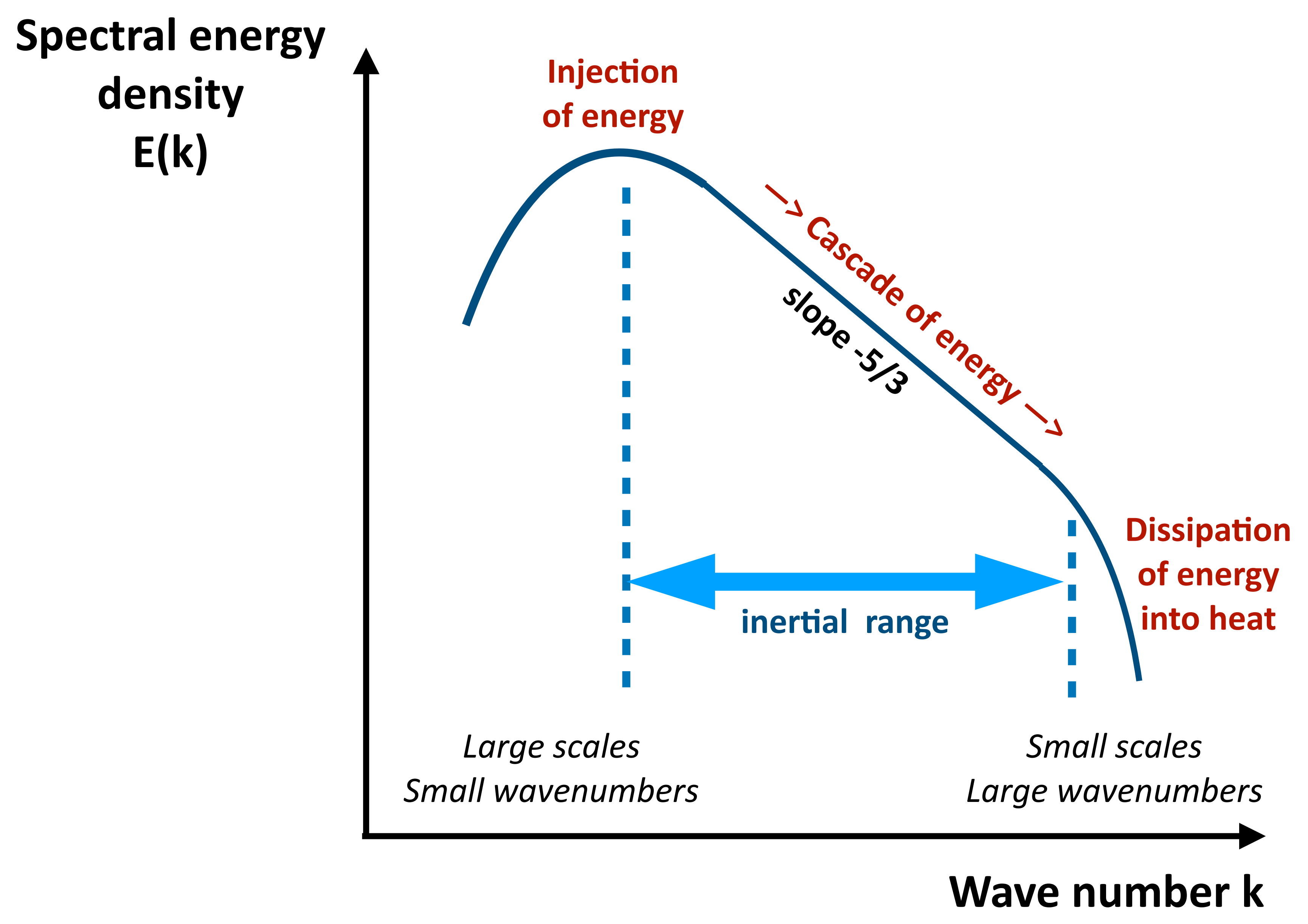
# Outline

- Historical remarks
- Two types of multifractal fields and how to analyze them
- Examples from the field of turbulence and other fields
- Conclusions, open questions and perspectives

# The energy cascade in turbulence

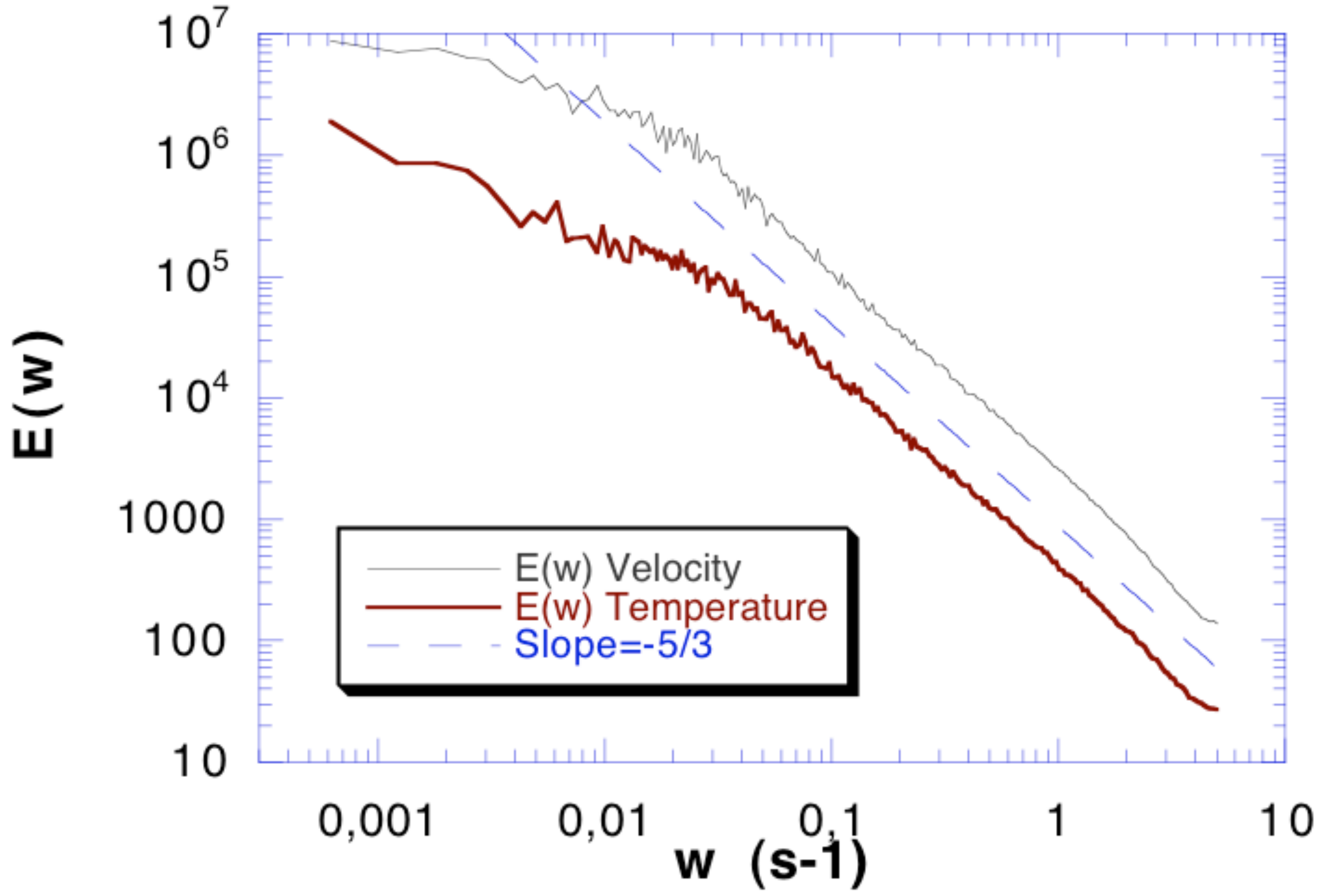
Richardson and Kolmogorov: energy cascade

- Richardson (1922): energy cascade from large to small scales
- Kolmogorov (1941), Obukhov (1941): dimensional analysis, leading to a scaling power spectra for velocity fluctuations in $k^{-5/3}$



# The energy cascade in turbulence

Experimental validation of the -5/3 law of Kolmogorov




$$E(k) \approx \varepsilon^{2/3} k^{-5/3}$$

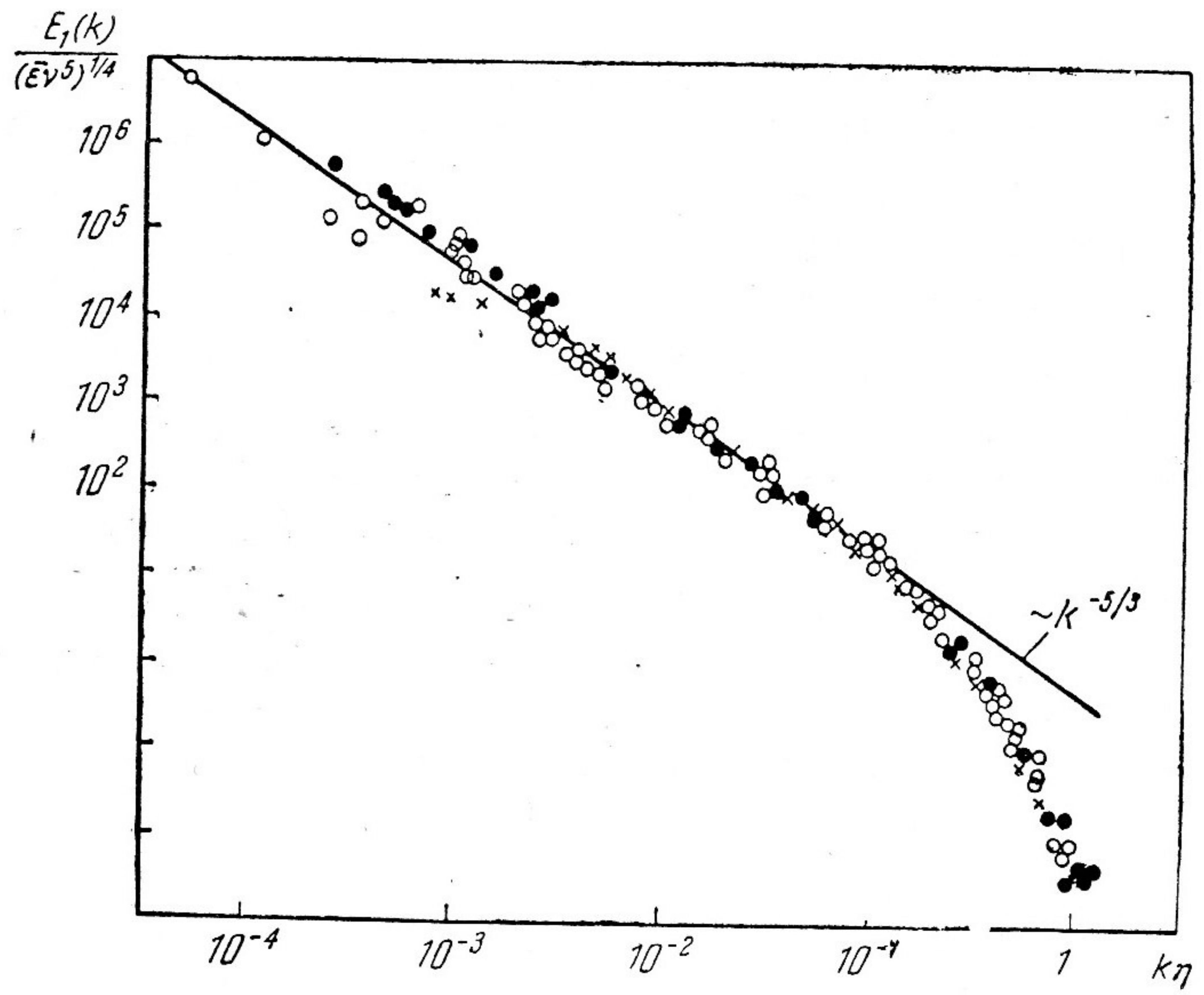


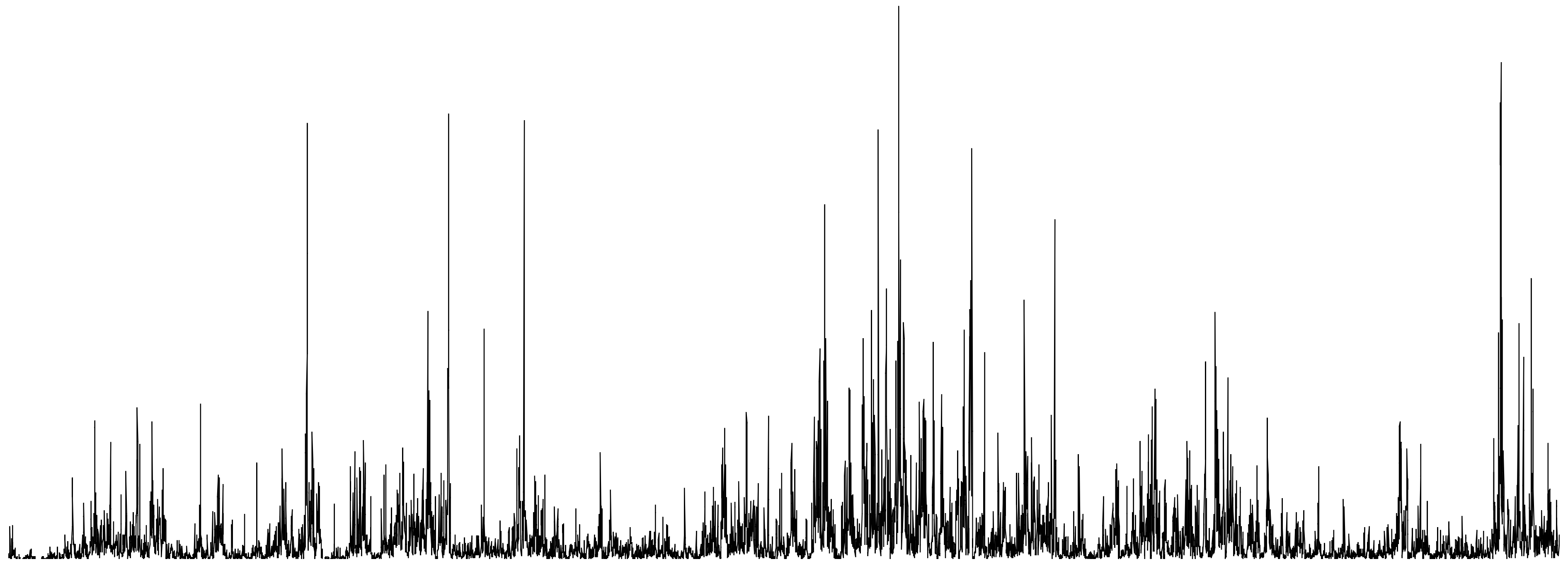

**Fig. 76** Normalized longitudinal velocity spectrum. ×—Sandborn and Marshall •—Grant, Stewart, and Moilliet; ○—Pond, Stewart, and Burling.

Checked in many situations since the 1960s

# The energy cascade in turbulence

## Intermittency

Batchelor and Townsend (1949): Experimental measurements of the dissipation: very strong fluctuations, are experimentally found, called « intermittency »



Obukhov (1962): locally averaged dissipation field, and assumption of lognormal fluctuations

* This assumption is not very restrictive as an approximate hypothesis since the distribution of any essentially positive characteristic can be represented by a logarithmically Gaussian distribution with correct values of the first two moments (see also Kolmogorov 1941*b*).

Then the average is

$$\tilde{\epsilon}(M_1, M_2) = \frac{1}{\frac{1}{6}\pi r^3} \int_{V_{M_1, M_2}} \epsilon \, dV.$$

Kolmogorov (1962): same hypothesis, and the variance depends on the log

$$\epsilon_r(\mathbf{x}, t) = \frac{3}{4\pi r^3} \int_{|\mathbf{h}| \leqslant r} \epsilon(\mathbf{x} + \mathbf{h}, t) \, d\mathbf{h}$$

averaged for a sphere of radius $r$, and in assuming that for large $L/r$ the logarithm of $\epsilon_r(\mathbf{x}, t)$ has a normal distribution. It is natural to suppose that the variance of $\log \epsilon_r(\mathbf{x}, t)$ is given by

$$\sigma_r^2(\mathbf{x}, t) = A(\mathbf{x}, t) + 9k \log L/r, \tag{2}$$

# The energy cascade in turbulence

Spikes have a spatial structure

Intermittency of the dissipation, with power-law fluctuations

$$B_{\epsilon\epsilon}(r) = <\epsilon(x+r)\epsilon(x)> = B <\epsilon^2> \left(\frac{L}{r}\right)^{\mu}$$

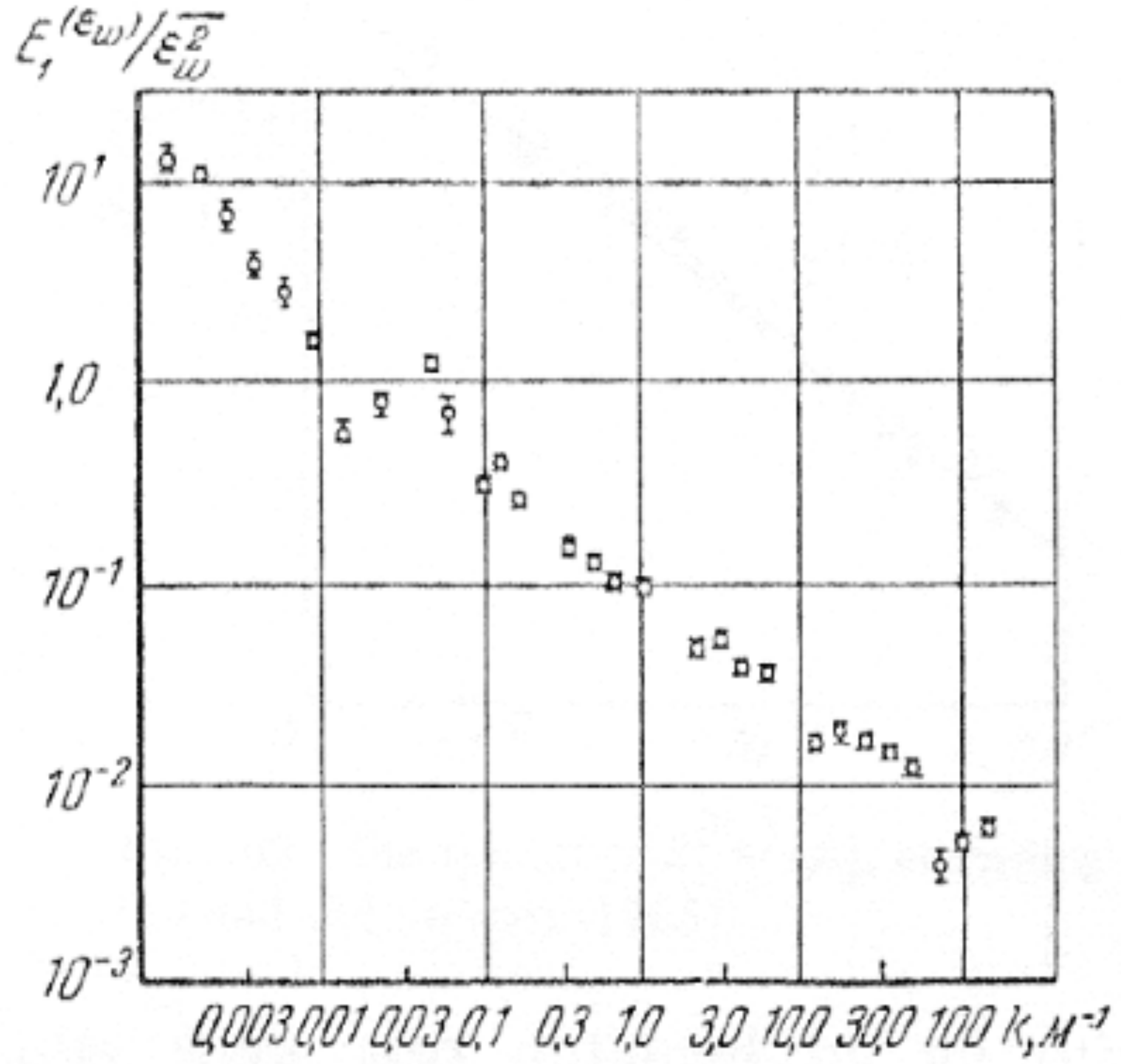


Fig. 102 The spectrum $E_1^{(\varepsilon_w)}(k)$, according to Gurvich and Zubkovskii (1963).

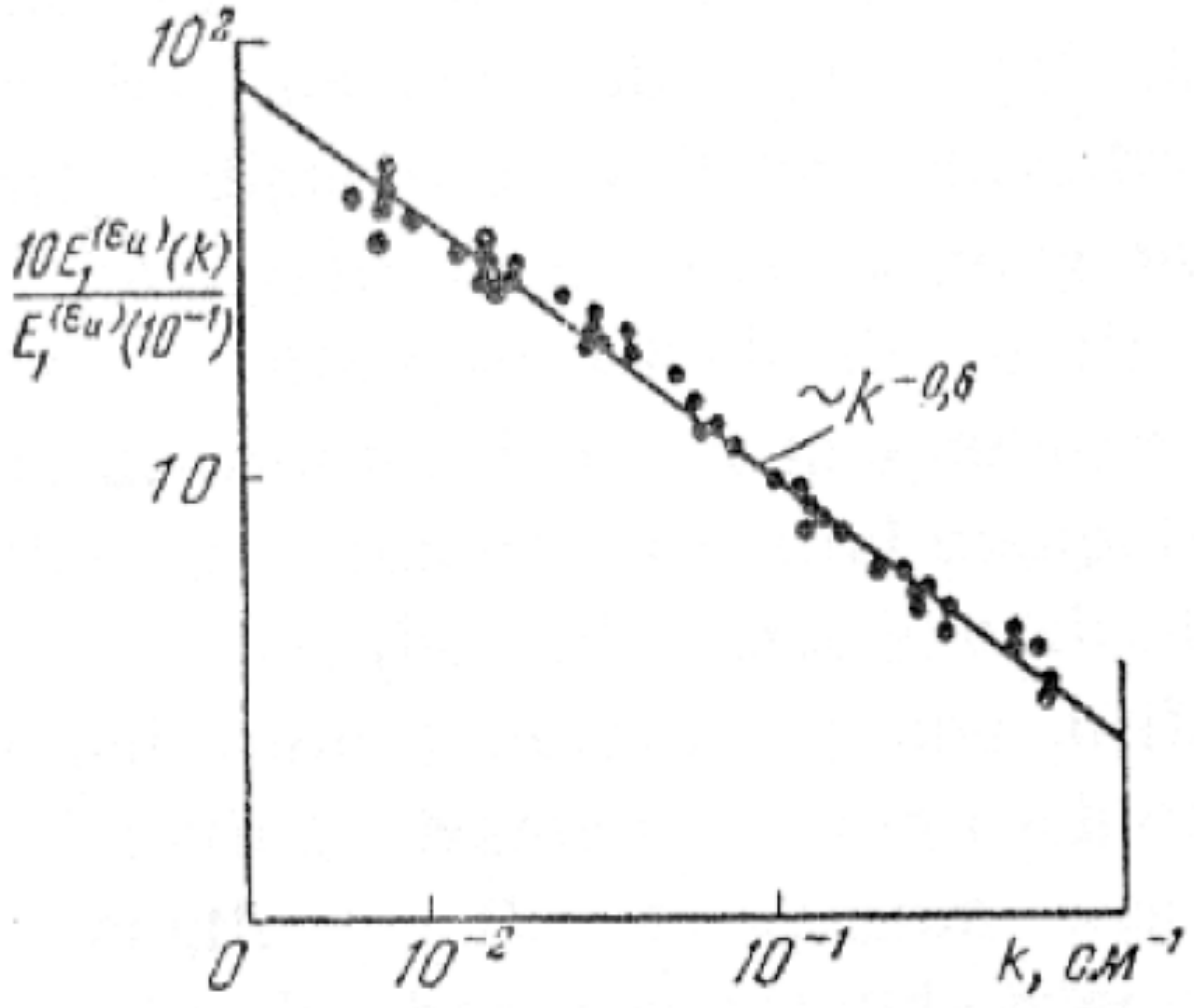


Fig. 103 The spectrum $E_1^{(\varepsilon_u)}(k)$, according to Pond and Stewart (1965).

Experimental results in Soviet Union (1963-1965): power-law correlation of the small scale dissipation

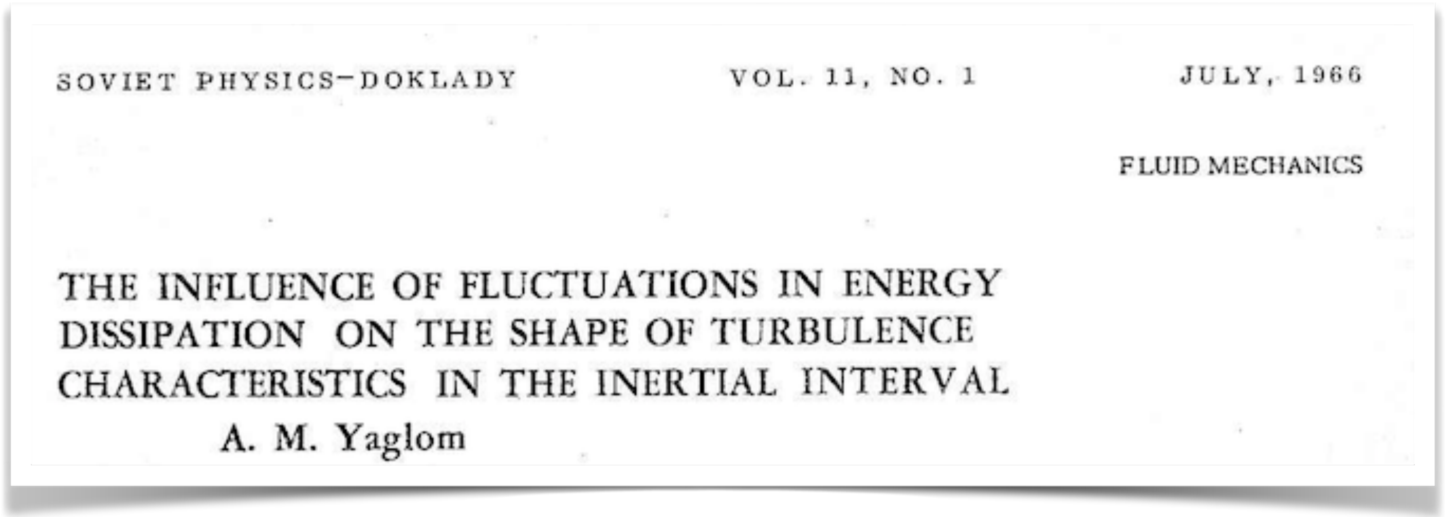
Gurvich and Zubkovskii (1963). Pond and Stewart (1965)

# The energy cascade in turbulence

## Yaglom's discrete multiplicative cascade

Yaglom (1966) recursive multiplicative cascade model:

- the first discrete cascade model in turbulence
- multifractal properties

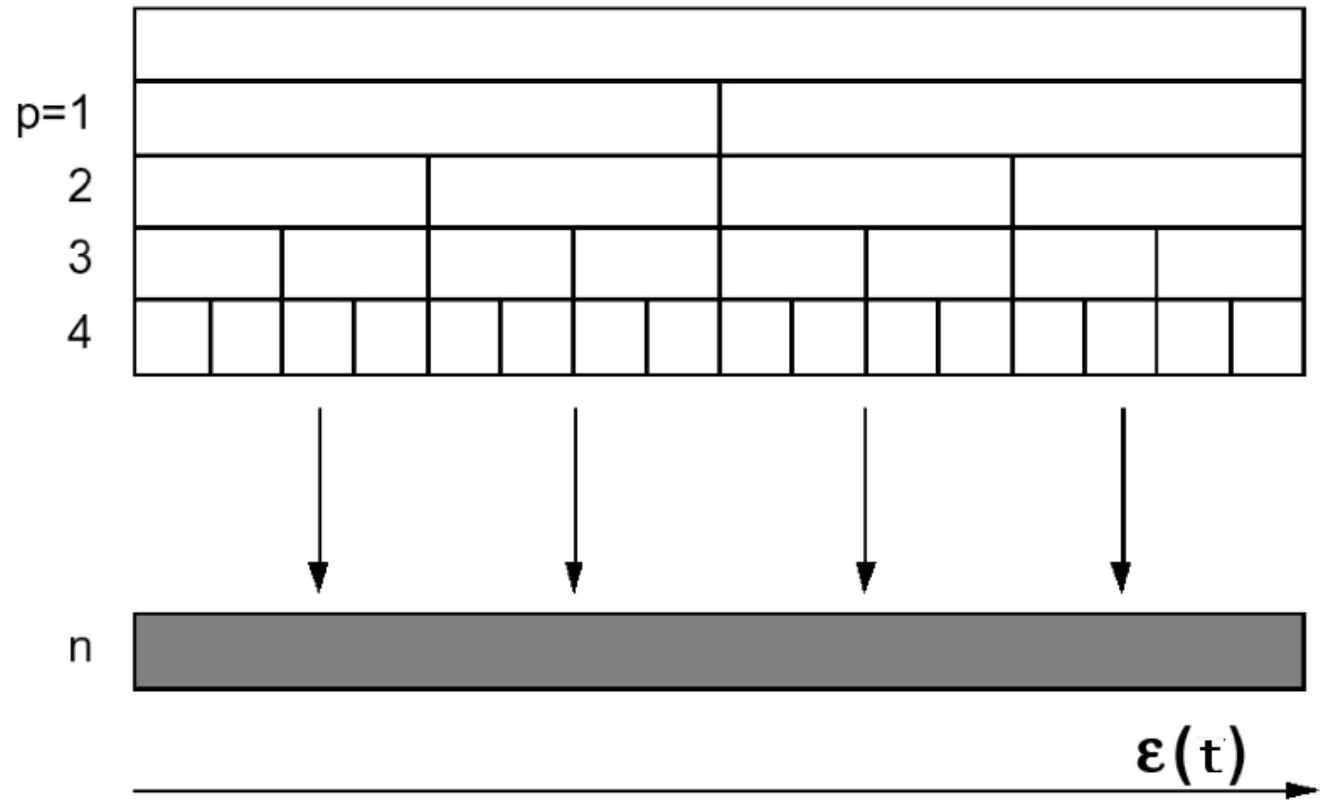


SOVIET PHYSICS–DOKLADY VOL. 11, NO. 1 JULY, 1966

FLUID MECHANICS

THE INFLUENCE OF FLUCTUATIONS IN ENERGY DISSIPATION ON THE SHAPE OF TURBULENCE CHARACTERISTICS IN THE INERTIAL INTERVAL

A. M. Yaglom




**Motivations**: Yaglom, as Kolmogorov's student, wanted to build a model compatible with:

- Kolmogorov's hypotheses
- long-range power-law correlations of epsilon as shown by experimental data

(i) generate lognormal statistics;

(ii) with power-law long-range correlations

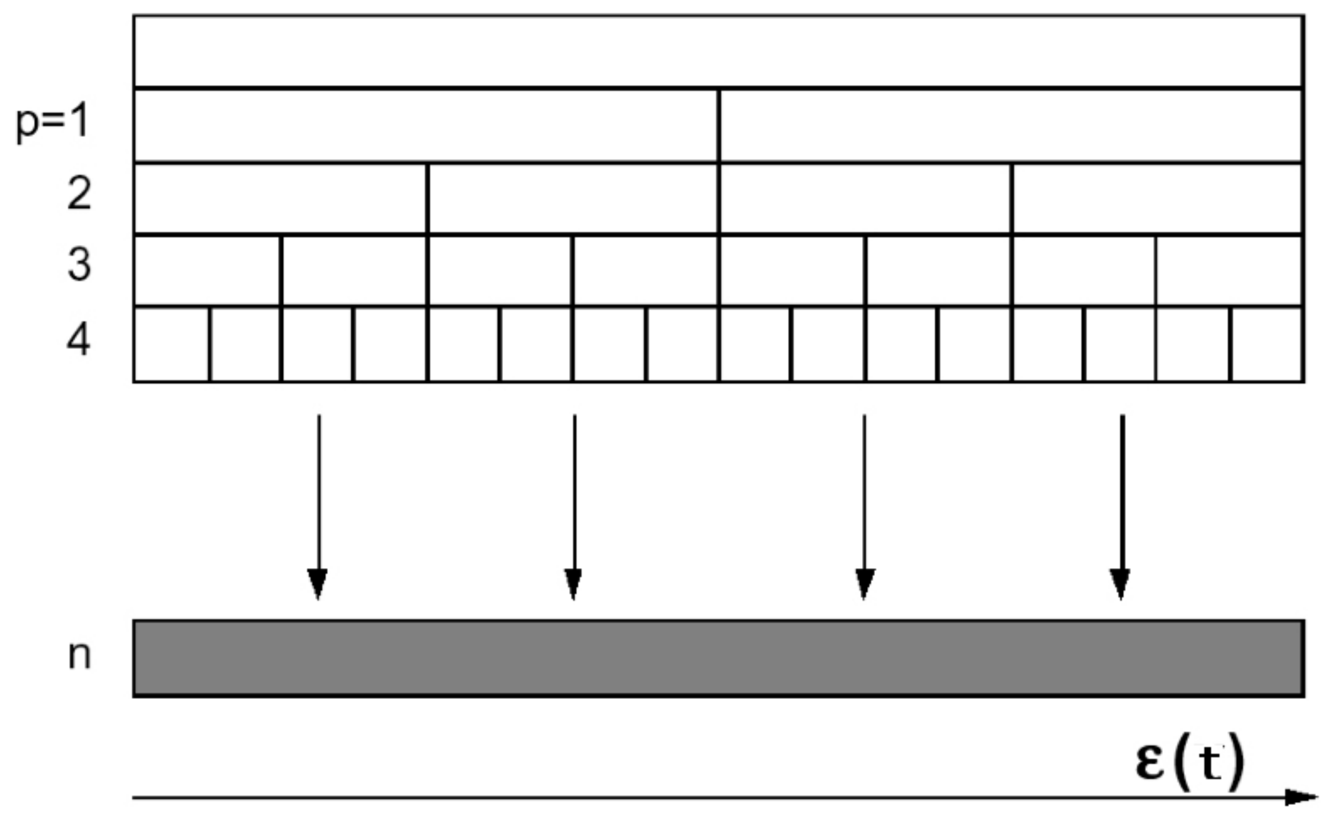
A generic model for multiplicative cascades still today.

Gives rise to multifractal statistics

# The energy cascade in turbulence

Yaglom's discrete multiplicative cascade: properties

$$\epsilon(x) = \prod_{i=1}^{n} W_{i,x}$$




### Scaling

K(q)=second characteristic function, or cumulant generating function, or scale invariant moment function

$$< \epsilon(x)^q > \approx \lambda^{K(q)}$$

$$\lambda = \frac{L}{\ell} = 2^n$$

$$K(q) = \log_2 < W^q >$$

### Power-law correlations

$\mu = K(2)$ intermittency parameter

$$< \epsilon(x)\epsilon(x+r) > \approx r^{-\mu}$$

### Logarithmic relations for the generator

$\gamma(x) = \log \epsilon(x)$ generator

$$< \gamma(x)\gamma(x+r) > \approx A - B \log r$$

log-correlation

$$\sigma_\gamma^2 = n\sigma_{\log W_i}^2 = \left(\frac{\sigma_{\log W_i}^2}{\log 2}\right) \log\left(\frac{L}{\ell}\right) = A' \log\left(\frac{L}{\ell}\right)$$

log law for the variance of $\log \epsilon$, as originally assumed by Kolmogorov (1962)

# A similar cascade model in two different fields

## Geology, for metal ore deposition

## Turbulence, for the dissipation

Krige (1951) — **lognormal statistics** — Obukhov (1962) Kolmogorov (1962)

Matheron (1962)
« regionalized variables »

**spatial average of a fluctuating quantity**

$$A_\ell(x) = \frac{1}{vol(B_\ell)} \int_{B_\ell(x)} \epsilon(x')dx'$$

Obukhov (1962)
Kolmogorov (1962)

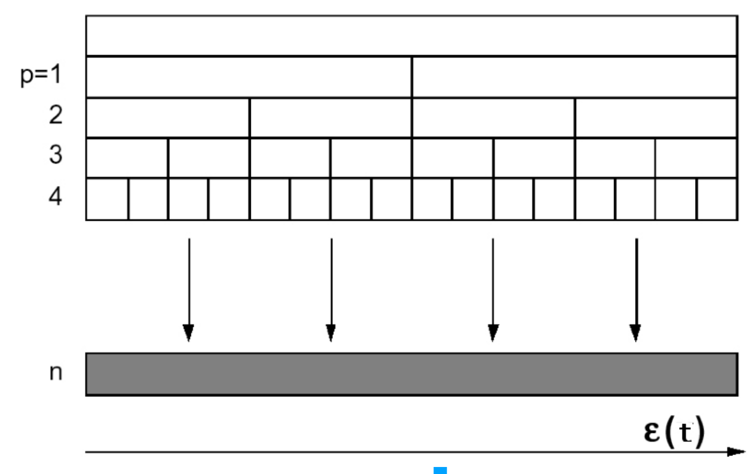
de Wijs (1951, 1953)

**Logarithmic relations for the generator**

$\gamma(x) = \log \epsilon(x)$ generator $\sigma_\gamma^2 = A' \log\left(\frac{L}{\ell}\right)$

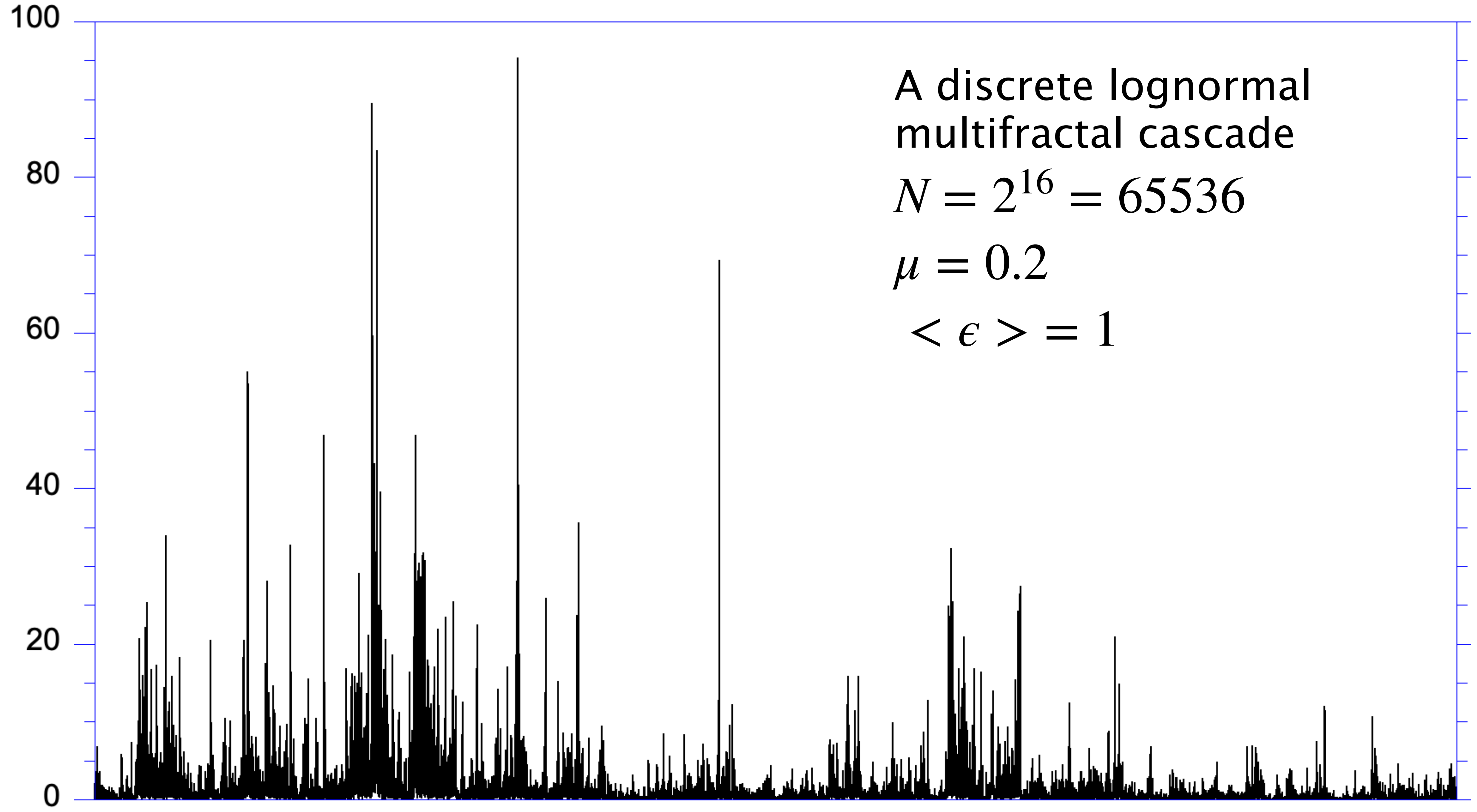
Kolmogorov (1962)
Yaglom (1966)

de Wijs (1951, 1953) — **Discrete embedded multiplicative model** — Yaglom (1966)

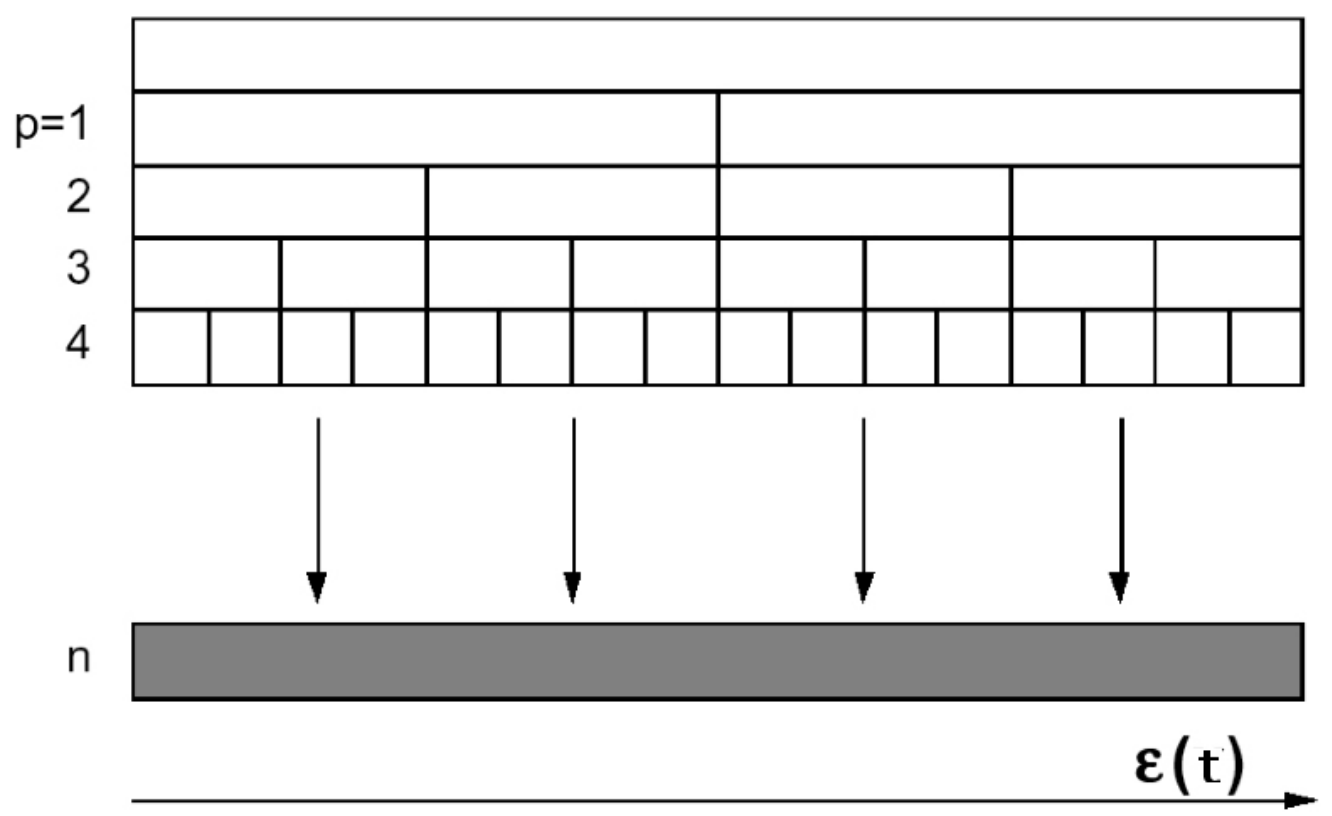



**See also:**
*Matheron's theory of regionalized variables*, Oxford University Press, 2019
Agterberg, *Geomathematics*, Springer, 2014

# Multifractal discrete cascades




$$\epsilon(x) = \prod_{i=1}^{n} W_{i,x}$$




**Intermittency**:

localized diverging pikes (singularities) with long-range correlations

## Discrete cascade models

Early proposal in geosciences: De Wijs (1951, 1953)

Early lognormal proposal in turbulence: Yaglom (1966)

Black-and-white $\beta$-model: Novikov and Stewart (1964); Frisch et al. (1978)

Random $\beta$-model: Benzi et al. (1984)

$\alpha$-model: Schertzer and Lovejoy (1984)

p-model: Meneveau and Sreenivasan (1987)

# Multifractal continuous cascades

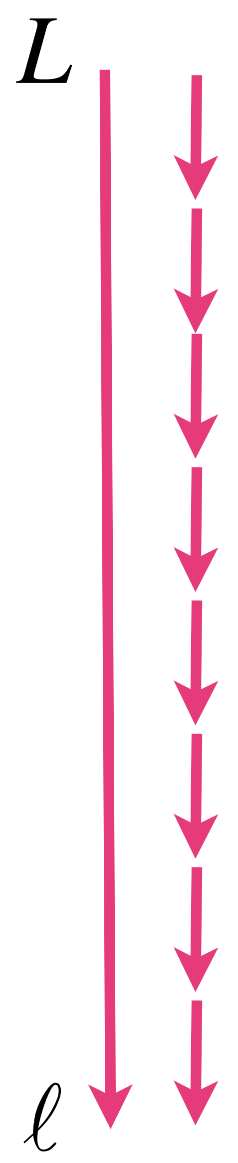
## semigroup property

Statistically, the cascade developed over a given scale ratio can be decomposed introducing any number of intermediary steps

Multiplication of random variables

$$\mathrm{X}_{L\to\ell} = \prod_{i=1}^{n} X_i$$

Each Xi is independent and has the same law: called «Independent and identically distributed – iid»




Addition of random variables

$$Y = \log X$$

$$Y_{L\to\ell} = \sum_{i=1}^{n} Y_i$$

Each Yi is independent and has the same law: called «Independent and identically distributed – iid»

«Infinitely divisible» law

## log-Infinitely Divisible cascade models

For a infinitely divisible random variable Y, for any integer n, we can write

$Y = \sum_{i=1}^{n} Y_i$, where the $Y_i$ are iid random variables

Each Yi is independent and has the same law: called «Independent and identically distributed – iid»

Since Novikov (1994) it has been recognized that for a continuous cascade (in scale), i.e. a cascade that can be indefinitely densified, the log of the process belongs to ID distribution.

This means that **continuous cascades have log-ID distributions**.

Examples of models which are log-ID:

- lognormal model (Kolmogorov 1962)
- log-stable model (Schertzer and Lovejoy, 1987; Kida 1991)
- log-Gamma model (Saito, 1992)
- log-Poisson model (She and Leveque 1994, She and Waymire 1995, Dubrulle 1995)

# Multifractal framework in turbulence

## For the dissipation

$$< \epsilon_\ell^q > \approx \ell^{-K(q)}$$ scale invariance of moments — $K(q)$ moment function

$$\Pr(\epsilon_\ell > \ell^{-\gamma}) > \approx \ell^{c(\gamma)}$$ scale invariance of singularities pdf — $c(\gamma)$ codimension function

$$< \epsilon_\ell^q > = \int \ell^{-q\gamma} dp(\gamma) \approx \ell^{\min_\gamma \{c(\gamma) - q\gamma\}}$$

Legendre transform
relation between moment function
and codimension function

$$K(q) = \max_\gamma \{q\gamma - c(\gamma)\}$$

## For the velocity

$$< |V(x+\ell) - V(x)|^q > \approx \ell^{\zeta(q)}$$ scale invariance of moments — $\zeta(q)$ moment function

$$\Pr(|V(x+\ell) - V(x)| > \ell^h) > \approx \ell^{c(h)}$$ scale invariance of singularities pdf — $c(h)$ codimension function

$$< |V(x+\ell) - V(x)|^q > = \int \ell^{qh} dp(h) \approx \ell^{\min_h \{c(h) + qh\}}$$

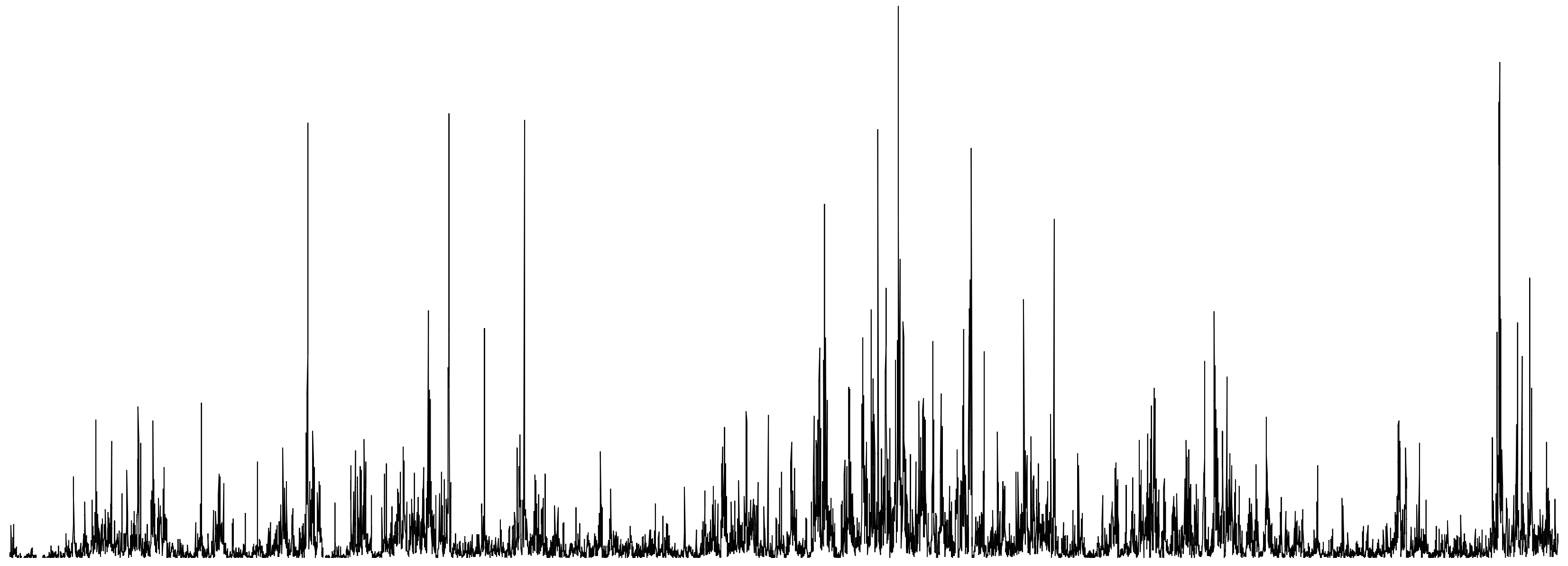
Legendre transform
relation between moment function
and codimension function
Parisi and Frisch (1985) (and
introduction of the word multifractal)

$$\zeta(q) = \min_h \{qh + c(h)\}$$

# Outline

- Historical remarks
- **Two types of multifractal fields and how to analyze them**
- Examples from the field of turbulence and other fields
- Conclusions, open questions and perspectives

# Two types of multifractals and two types of analysis



Singular measure; positive values with localized pikes

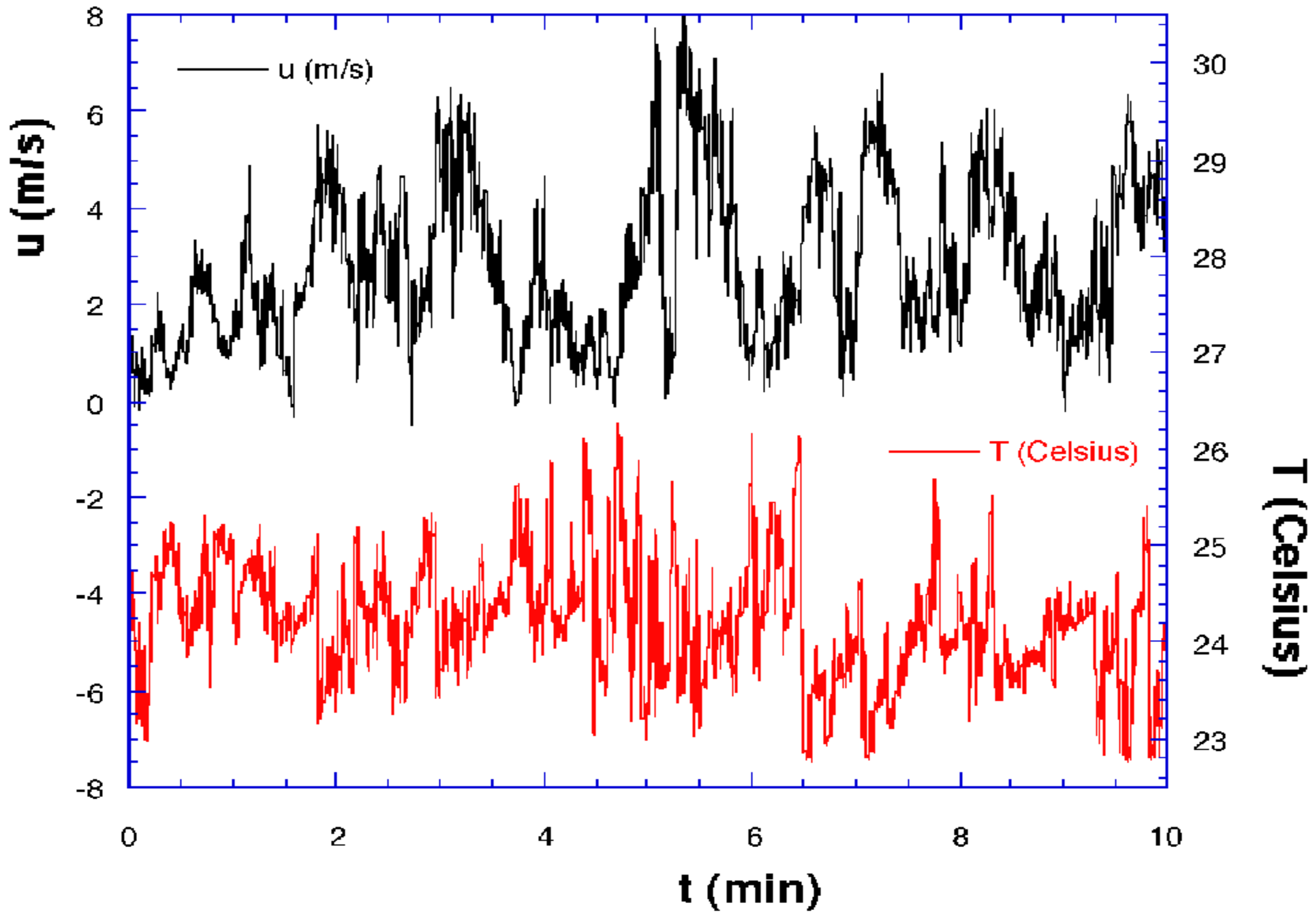
Directly produced by a multiplicative process

Ex: dissipation in turbulence

Scaling properties through coarse-graining: volume average of the small-scale singular field

$$\epsilon_\ell(x) = \frac{1}{vol(B_\ell)} \int_{B_\ell(x)} \epsilon(x')dx'$$

$$< \epsilon_\ell(x)^q > = C_q \ell^{-K(q)}$$




Stochastic processes with stationary increments (also called « multiaffine » or « non-stationary multifractals »)

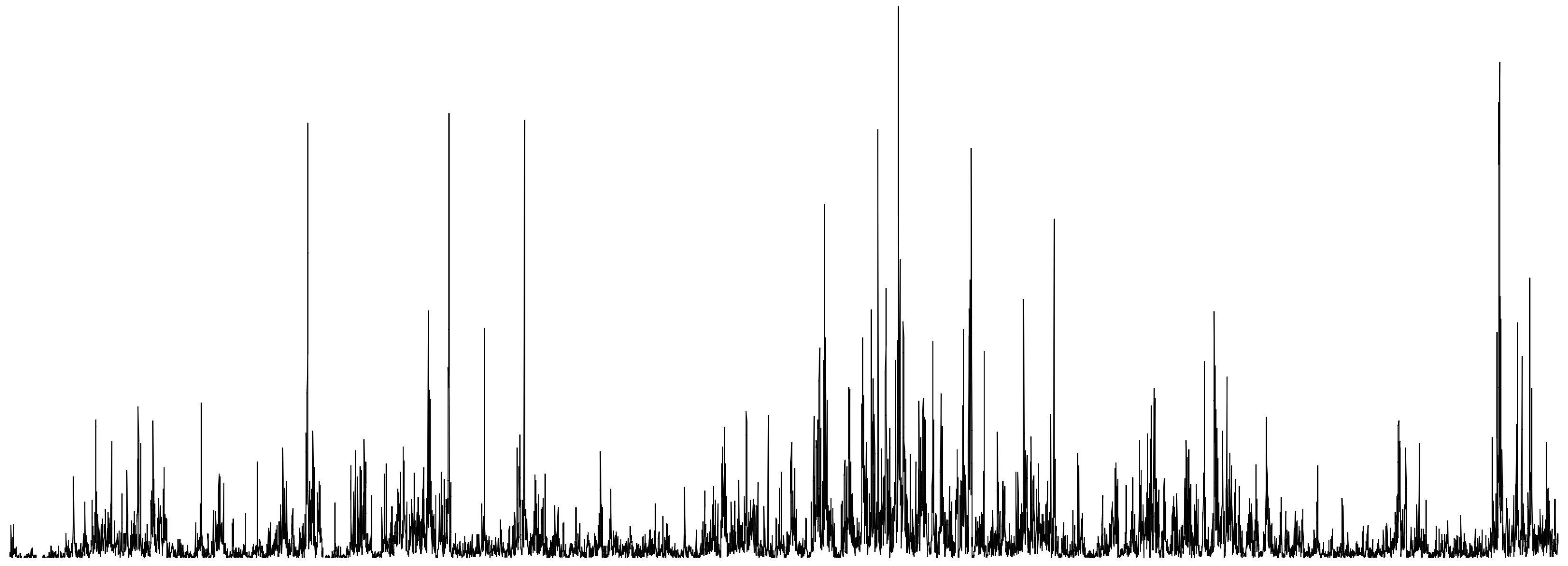
Mixture of additive and multiplicative processes

Ex: velocity, passive scalars, in turbulence

Scaling properties through increments, or convolution with a kernel (wavelets) that suppresses a local trend

$$< |X(x+\ell) - X(x)|^q > = C_q \ell^{\zeta(q)}$$

# Two types of multifractals and two types of analysis



Singular measure; positive values with localized pikes

Directly produced by a multiplicative process

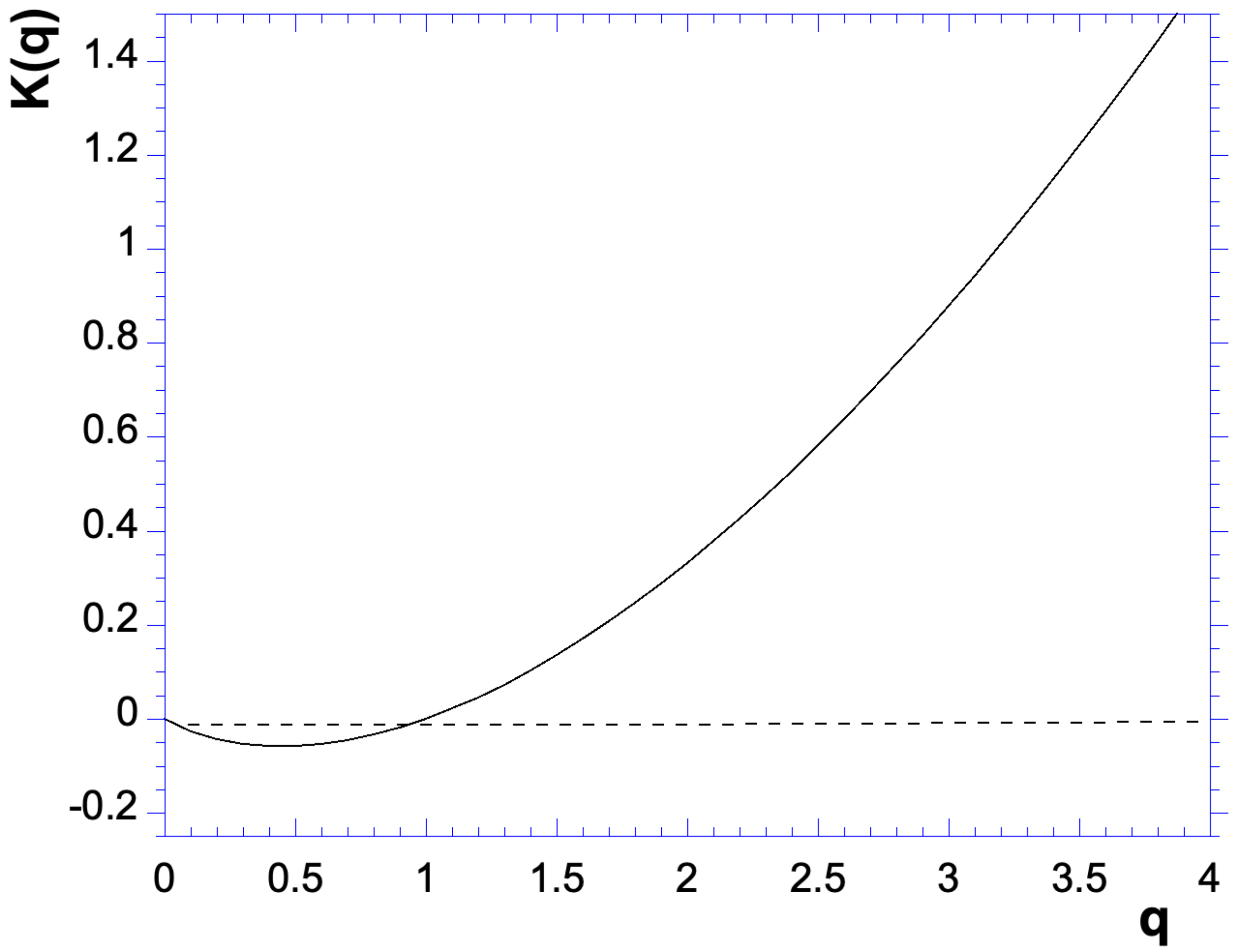
Ex: dissipation in turbulence

Scaling properties through coarse-graining: volume average of the small-scale singular field

$$\epsilon_\ell(x) = \frac{1}{vol(B_\ell)} \int_{B_\ell(x)} \epsilon(x')dx'$$

$$< \epsilon_\ell(x)^q > = C_q \ell^{-K(q)}$$

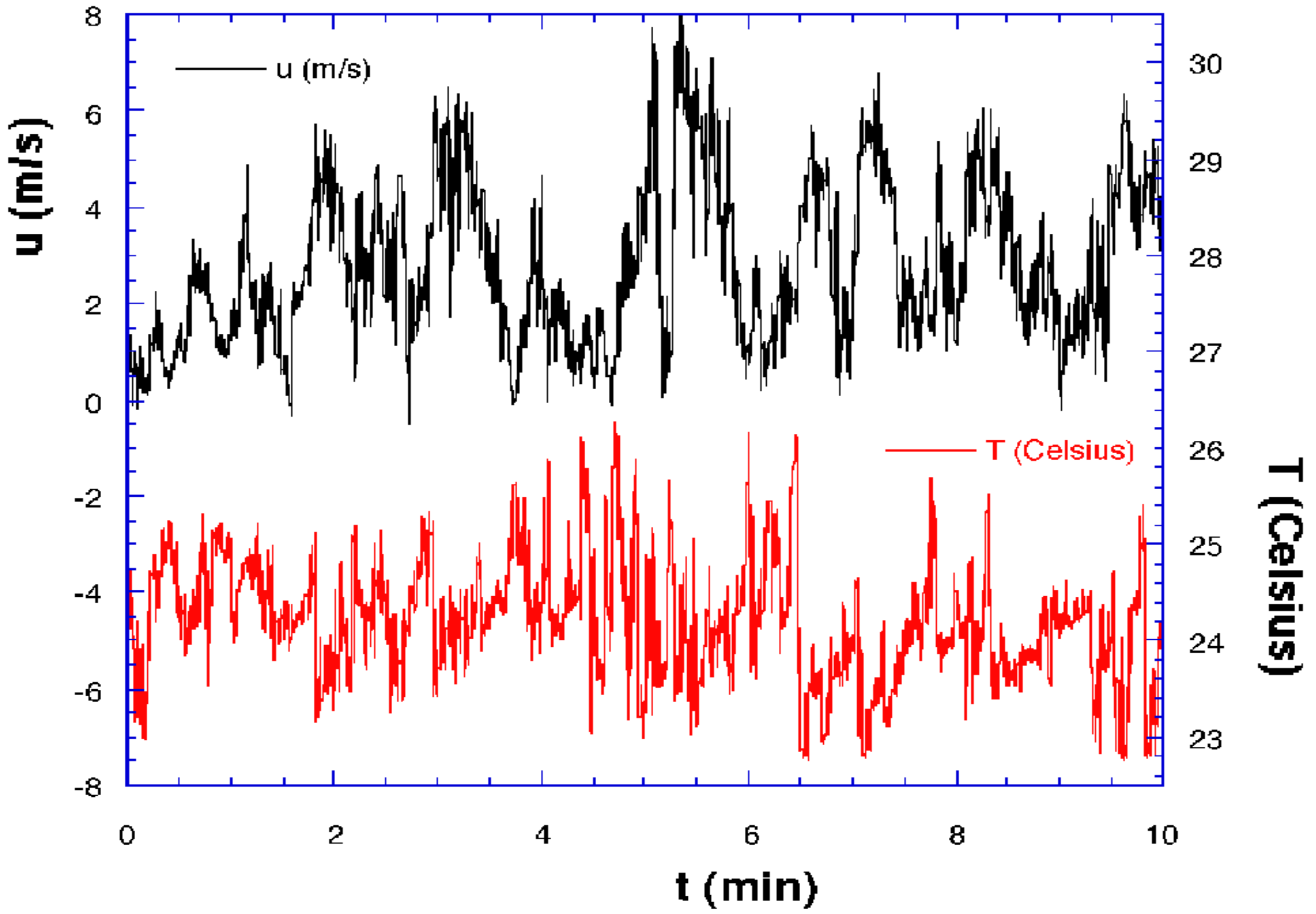
Since local average does not change the mean, scaling through coarse graining verifies $< \epsilon_\ell(x) >$ = constant, hence $K(1) = 0$




Examples of analytical expressions:

- log-Poisson model: $K(q) = c[(1-\beta)q - 1 + \beta^q]$ (with $c > 0$ and $0 < \beta < 1$)
- log-stable model: $K(q) = \dfrac{C_1}{\alpha - 1}\left(q^\alpha - q\right)$ (with $C_1 > 0$ and $0 < \alpha \leq 2$)

# Two types of multifractals and two types of analysis




Stochastic processes with stationary increments (also called « multiaffine » or « non-stationary multifractals »)

Mixture of additive and multiplicative processes

Ex: velocity, passive scalars, in turbulence

Scaling properties through increments, or convolution with a kernel (wavelets) that suppresses a local trend

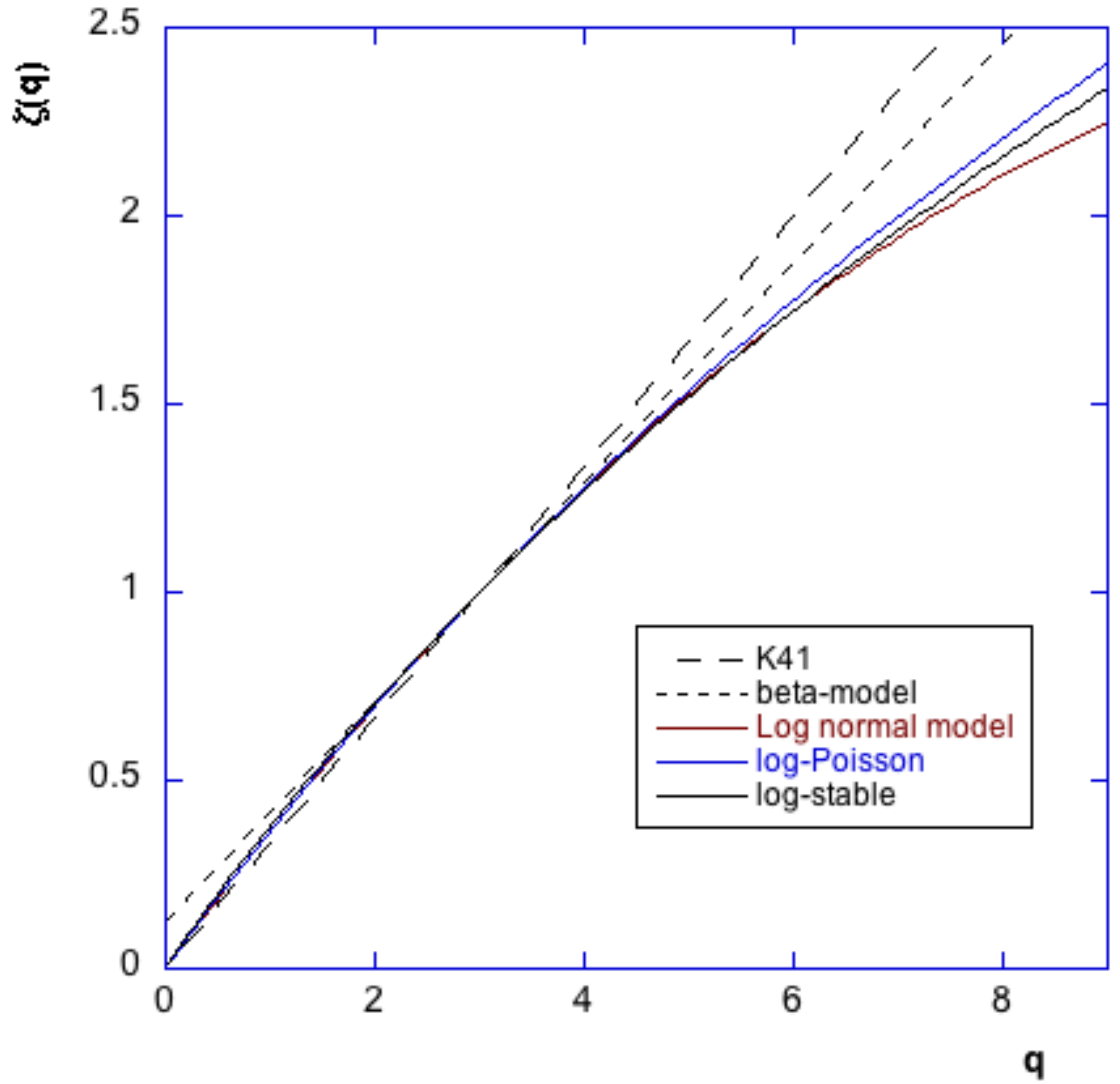
$$< |X(x+\ell) - X(x)|^q > = C_q \ell^{\zeta(q)}$$

Index of non-stationarity: $H = \zeta(1) \neq 0$

The curve $\zeta(q)$ is the sum of a linear trend and a nonlinear correction.

**Examples**:

- for the lognormal model, the correction is quadratic $\zeta(q) = qH - \dfrac{C_1}{\alpha - 1}\left(q^2 - q\right)$
- for the log-stable model, the correction is a power-law $\zeta(q) = qH - \dfrac{C_1}{\alpha - 1}\left(q^\alpha - q\right)$
- for the log-Poisson model, the correction is an exponential $\zeta(q) = qH - c[(1-\beta)q - 1 + \beta^q]$



# Outline

- Historical remarks
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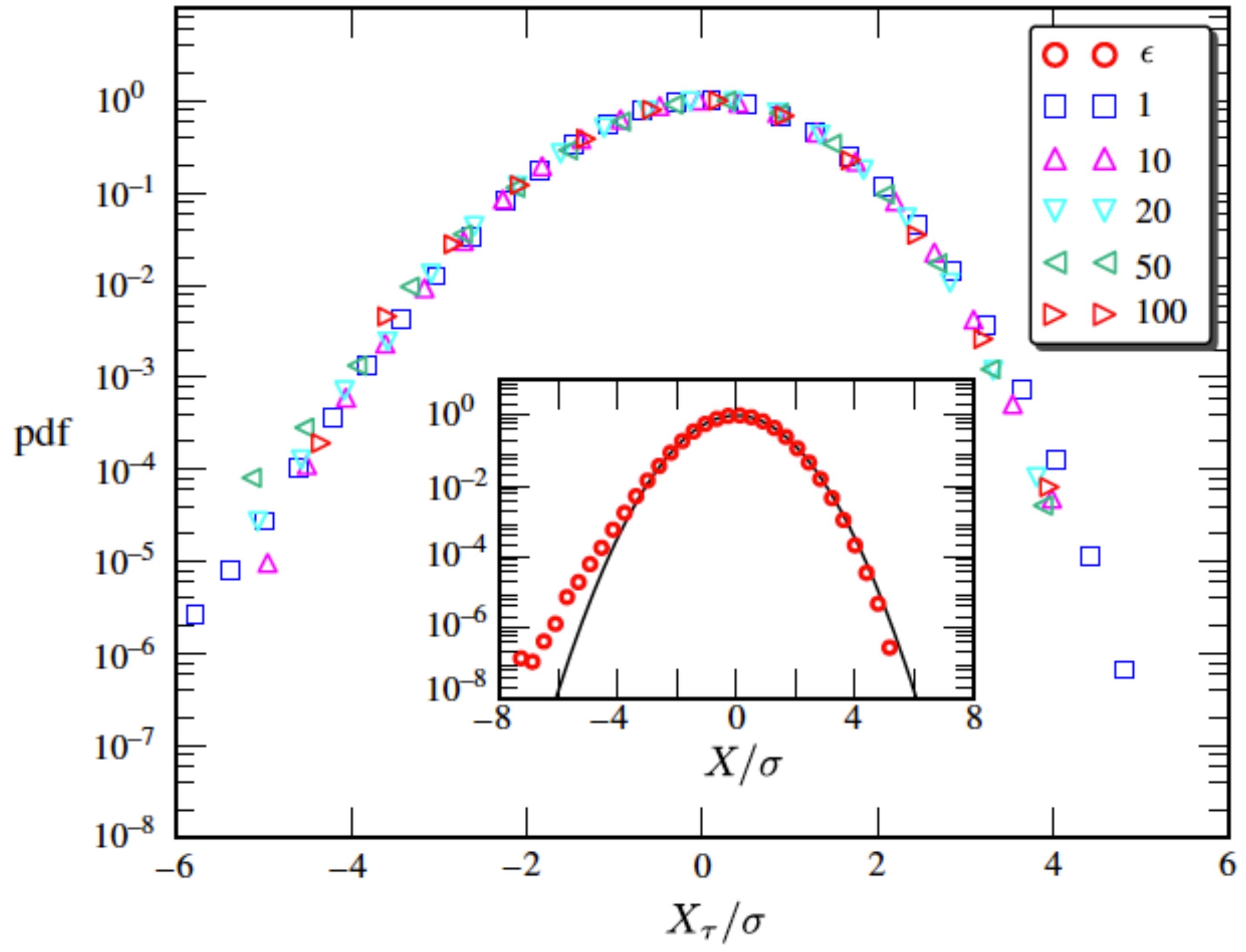
- **Examples from the field of turbulence and other fields**
- Conclusions, open questions and perspectives

# Examples: multifractal measures

**Dissipation in turbulence**: Lagrangian data from a Direct Numerical Simulation




PDF of log-dissipation

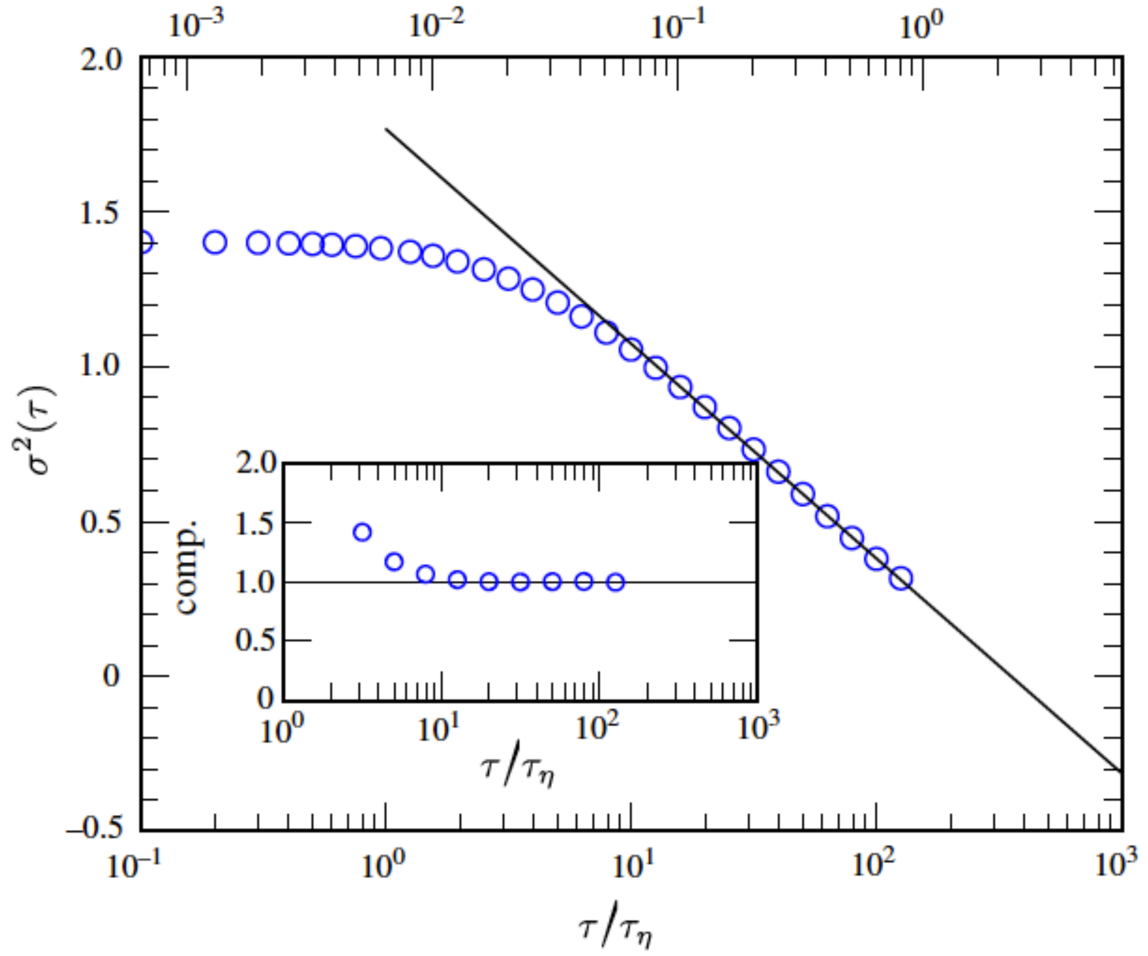
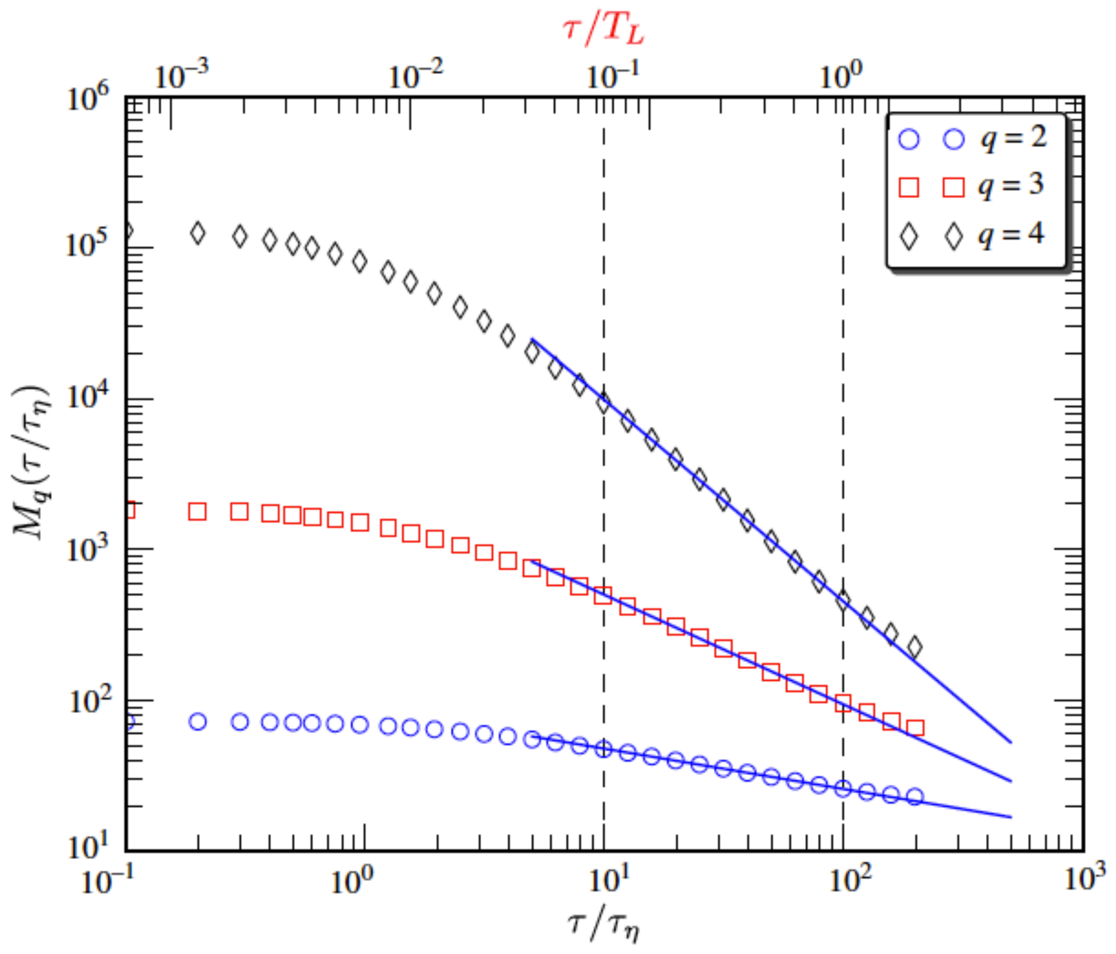
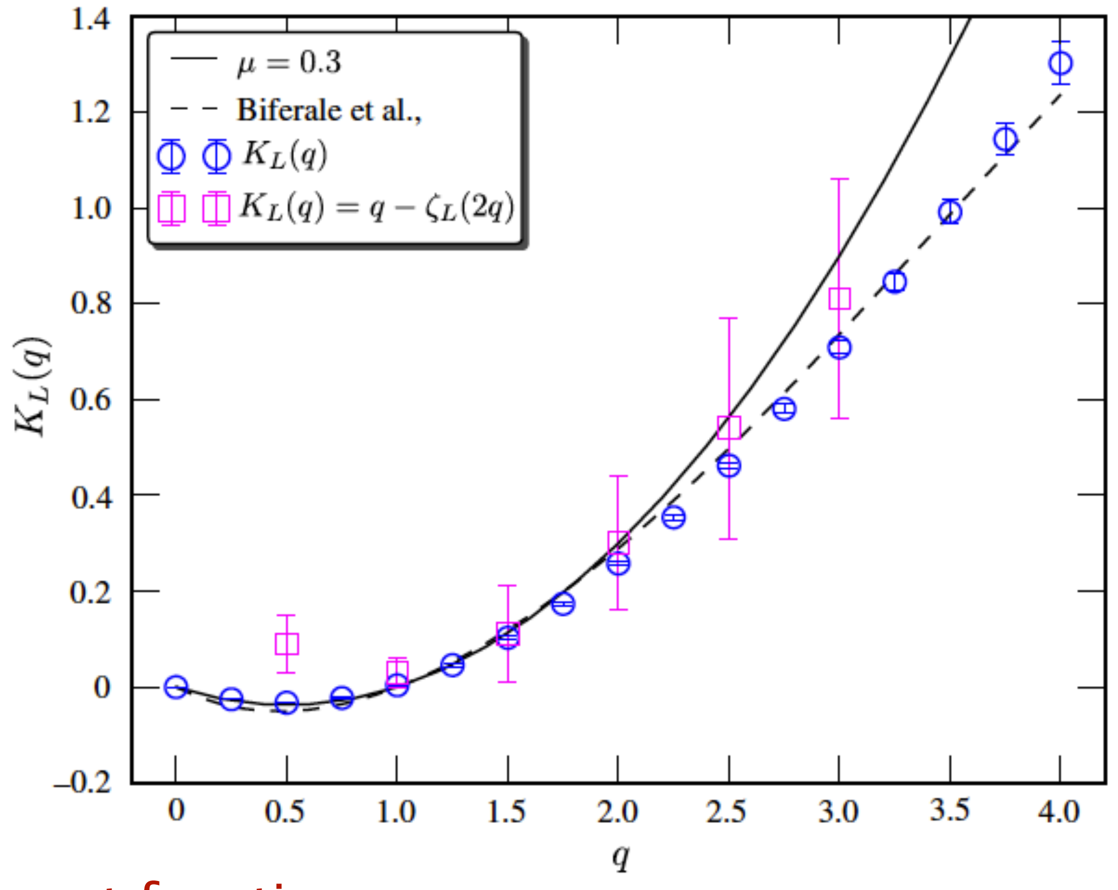



FIGURE 4. Measured variance $\sigma^2(\tau)$ (∘) of $\ln(\epsilon_\tau)$ (∘). A log-law is observed with a scaling exponent $\beta = 0.30 \pm 0.01$ on the range $10 < \tau/\tau_\eta < 100$. The inset shows the compensated curve with fitted parameters to emphasize the log-law. The statistical error of $\beta$ is estimated as in figure 2.

log–law for the variance of the log–dissipation



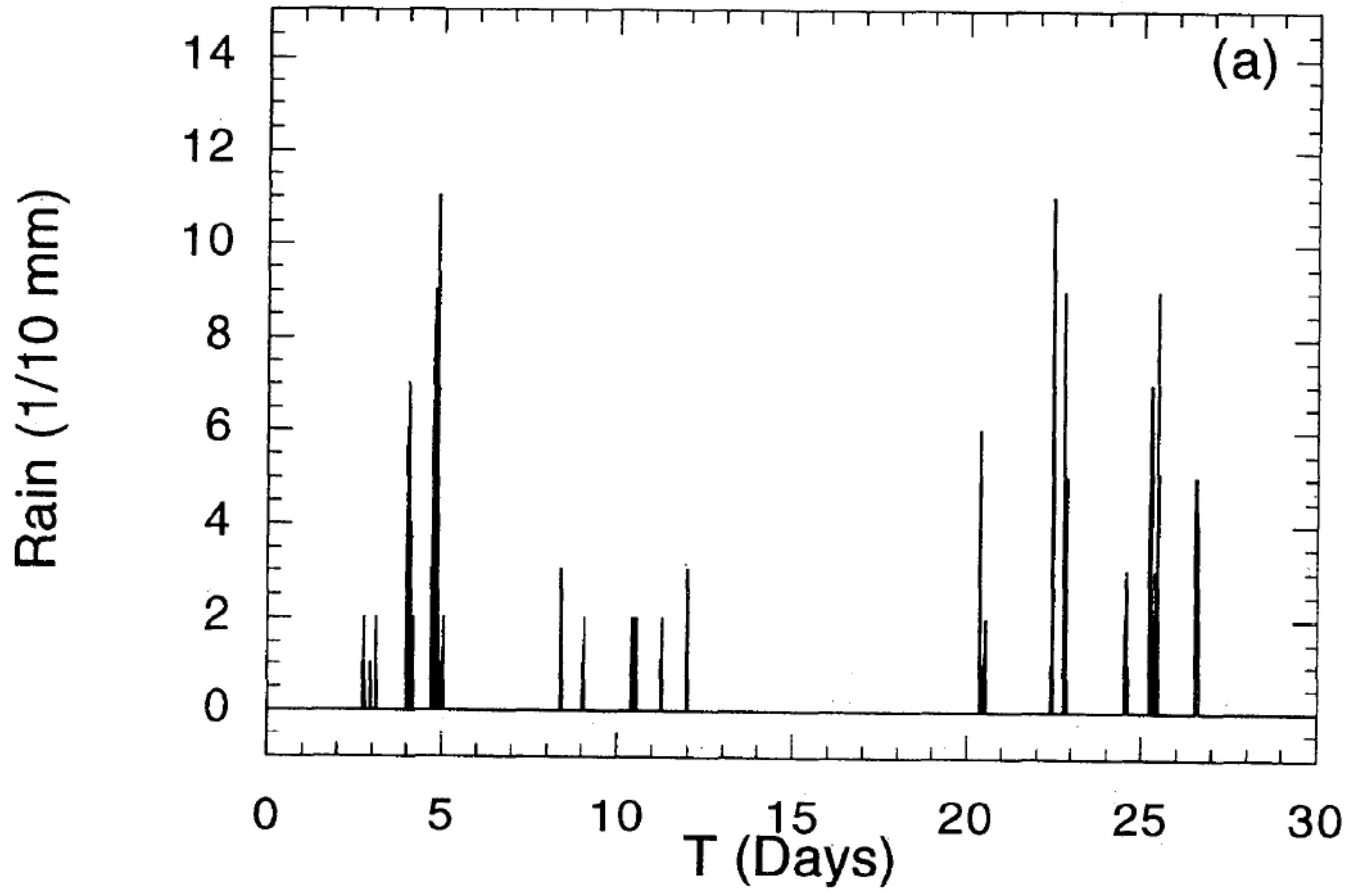




Scaling and moment function

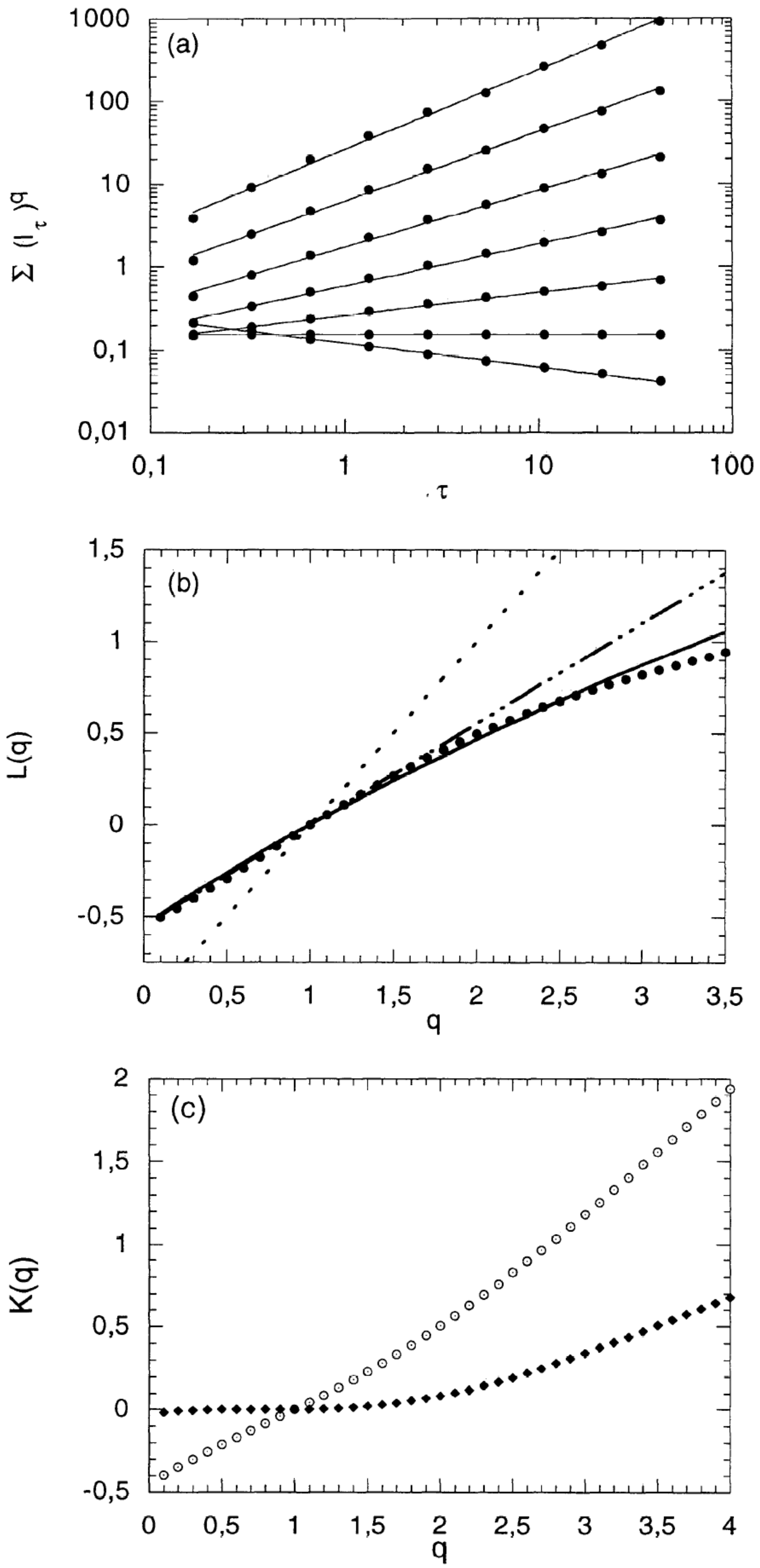
Huang Y., F. G Schmitt, Lagrangian cascade in three-dimensional homogeneous and isotropic turbulence, *Journal of Fluid Mechanics* 741, R2, 2014

# Examples: multifractal measures

**Rainfall**: many zeroes. Multifractal cascades with a fractal support




Data: intermittency with zero values

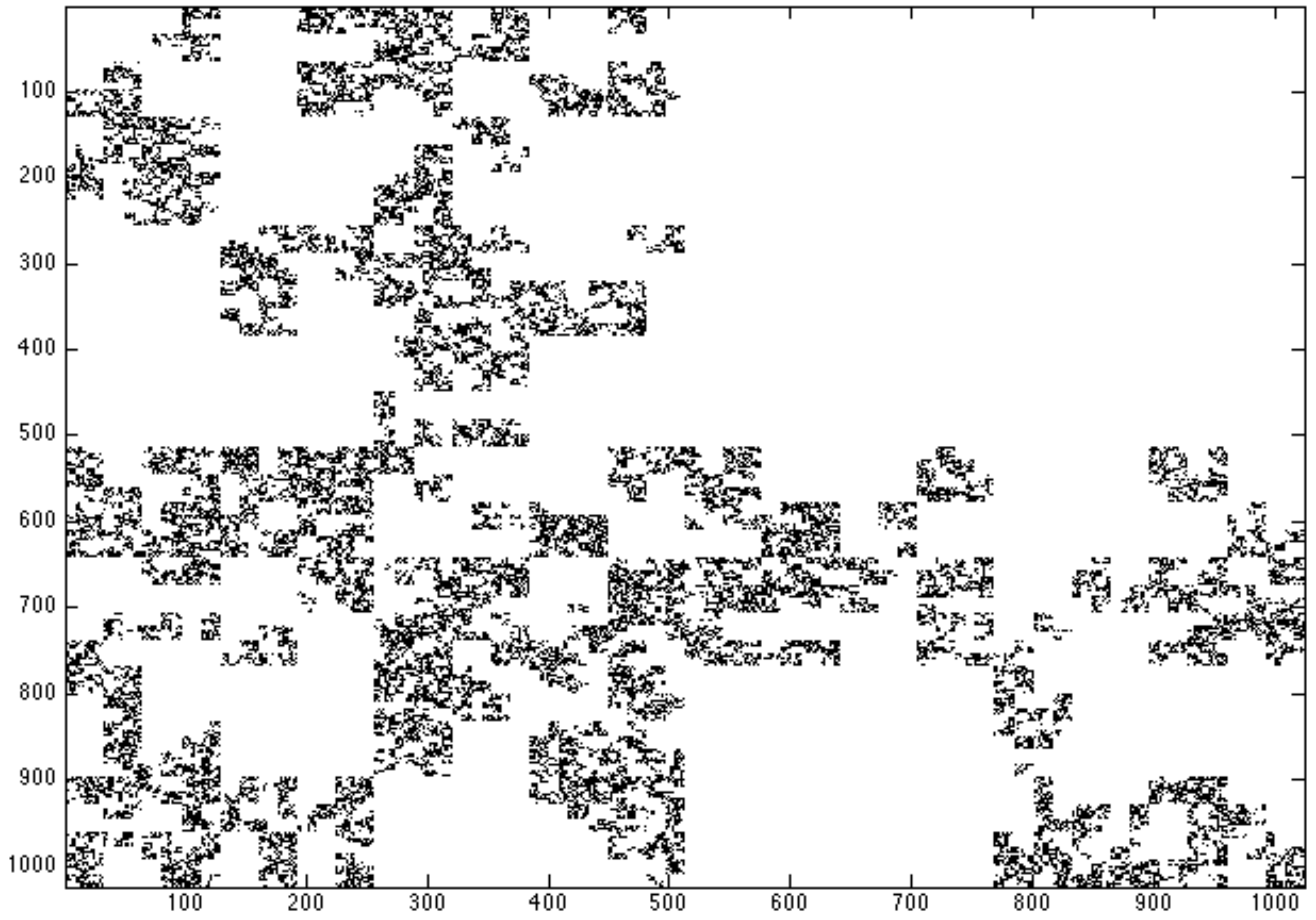



Scaling and moment function

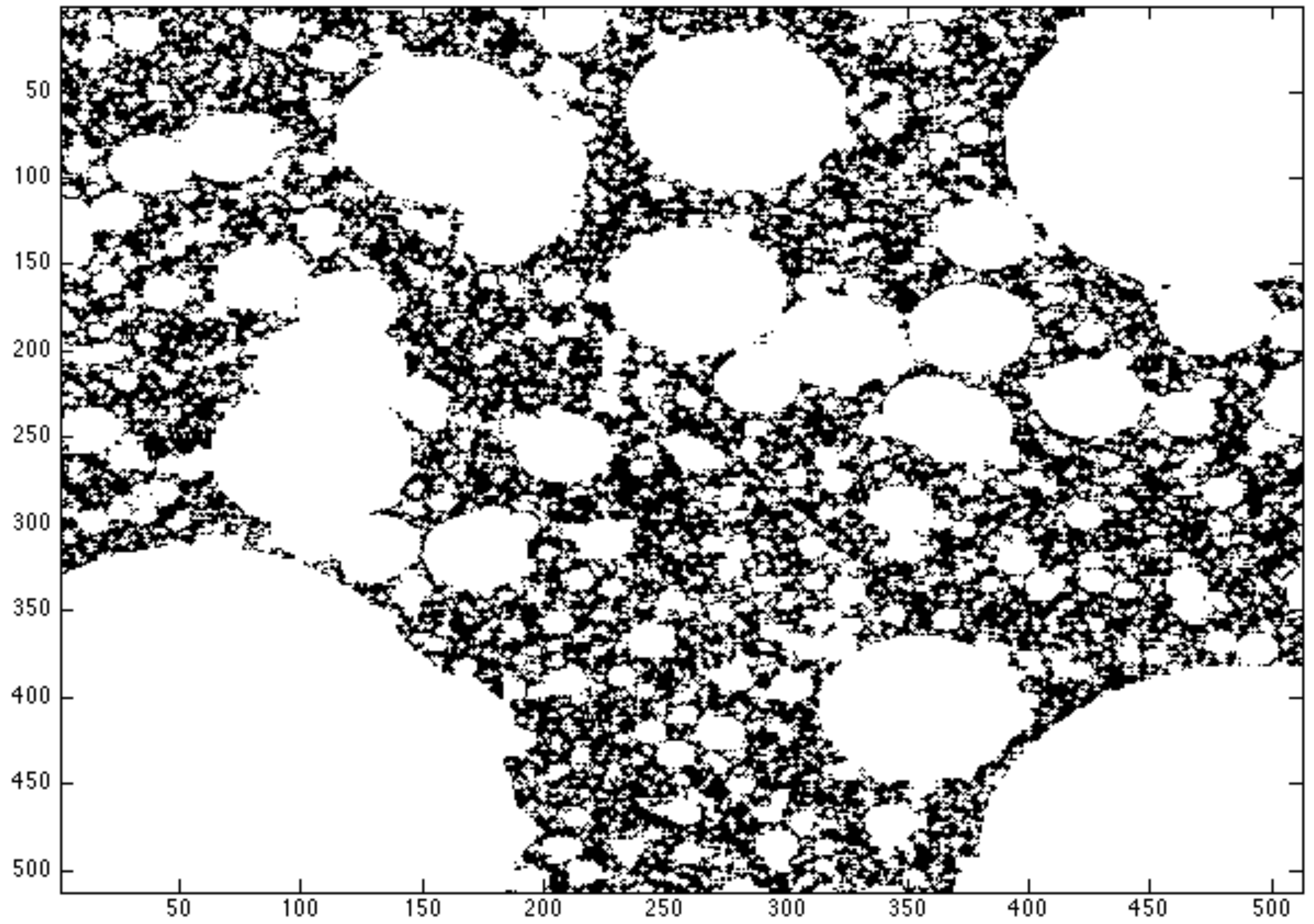
Schmitt, F., S. Vannitsem, and A. Barbosa: Modeling of rainfall time series using two-state renewal processes and multifractals, *Journal of Geophysical Research,* 103, vol. D18, 23181-23193, 1998

# Examples: multifractal measures

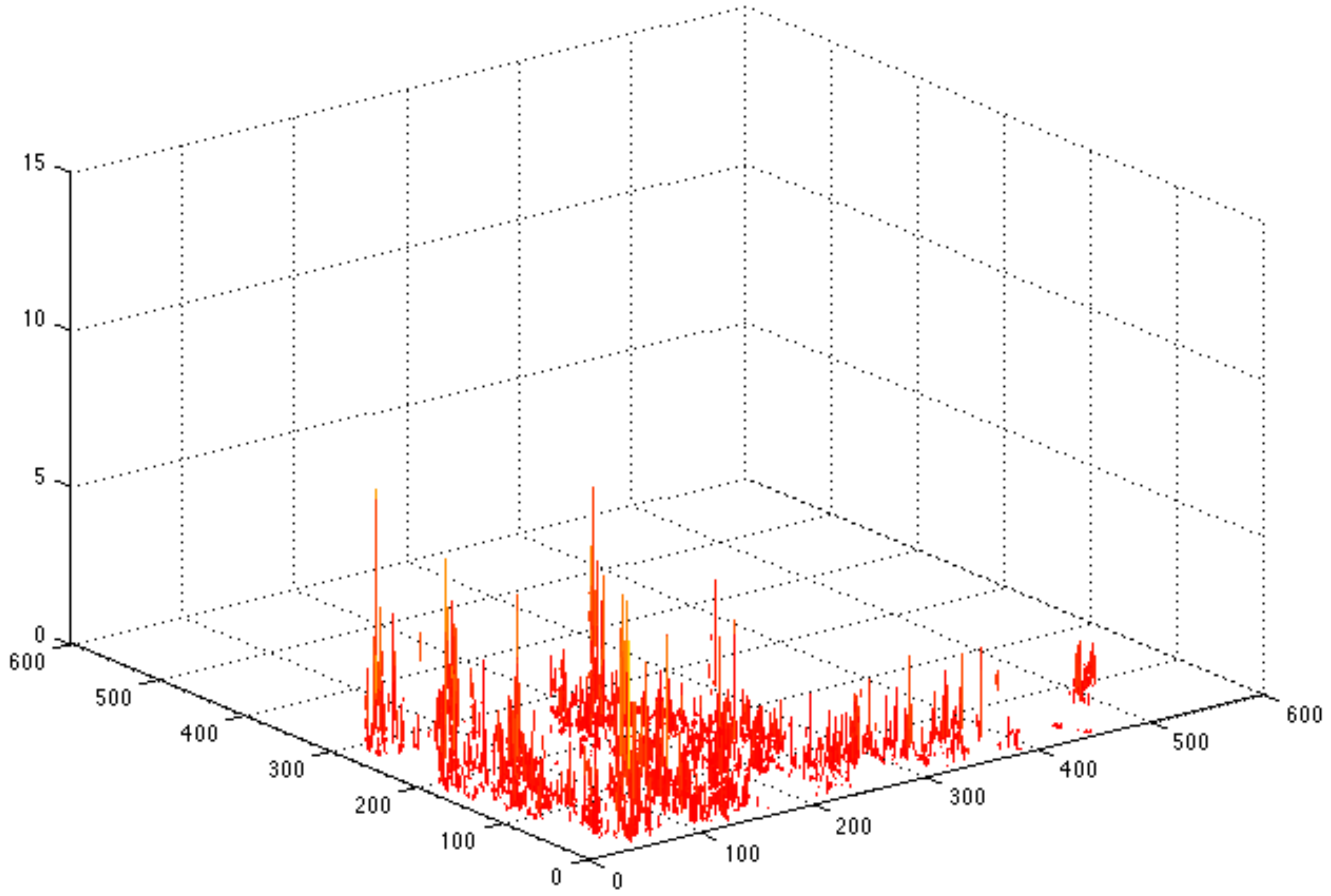
## **Rainfall**: a multiplicative model with zero values: a continuous $\beta$–multifractal model



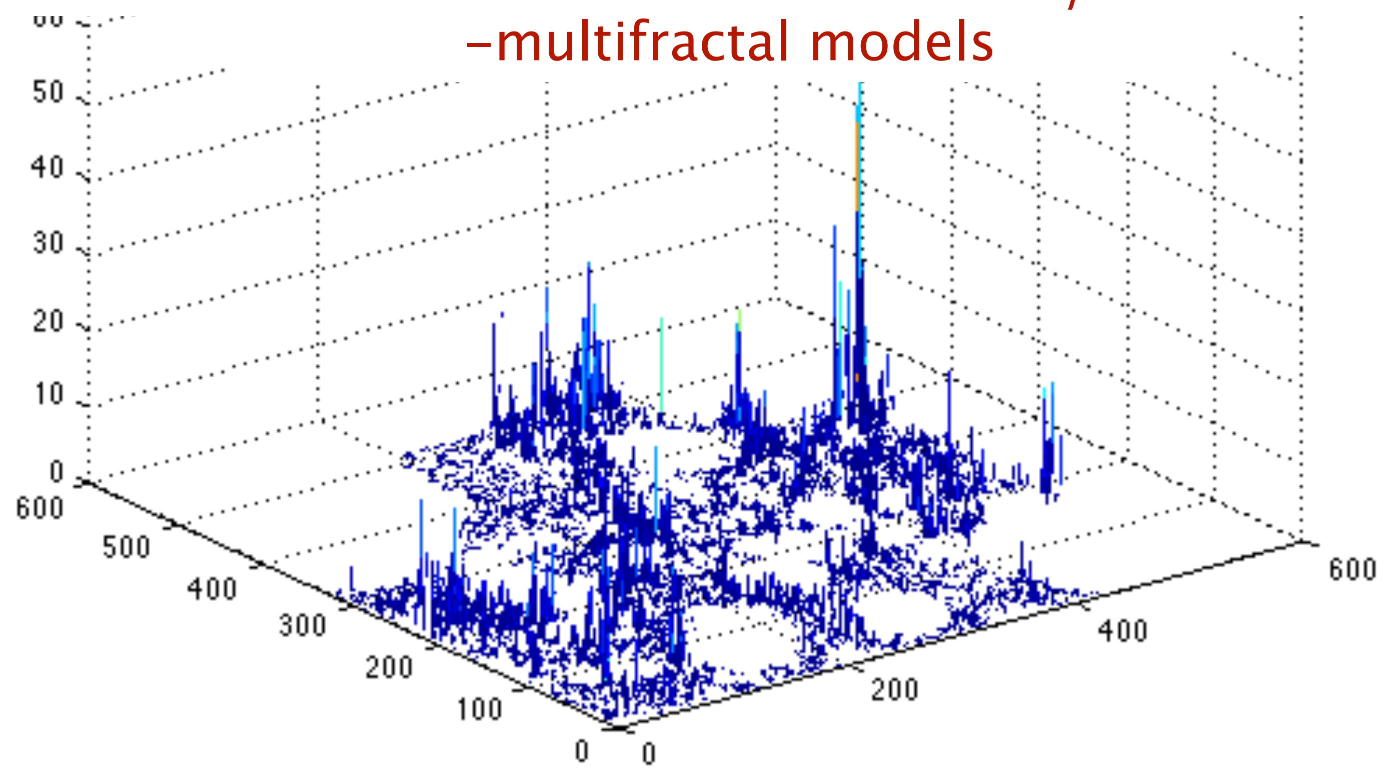

a discrete simulation of a $\beta$–model

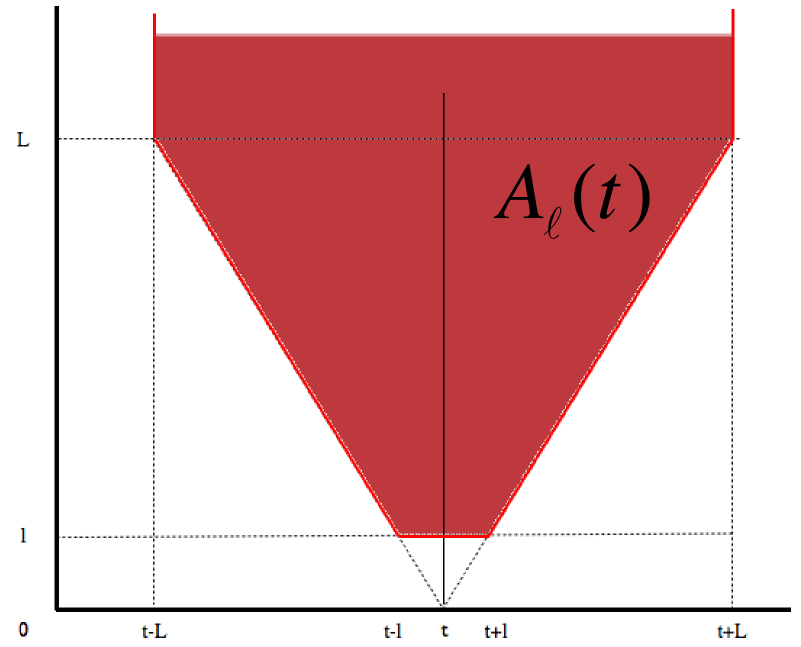



a continuous simulation of a $\beta$–model




continuous simulations of $\beta$ –multifractal models




$$\boxed{\varepsilon(t) = C.N\big(A_\ell(t)\big)}$$

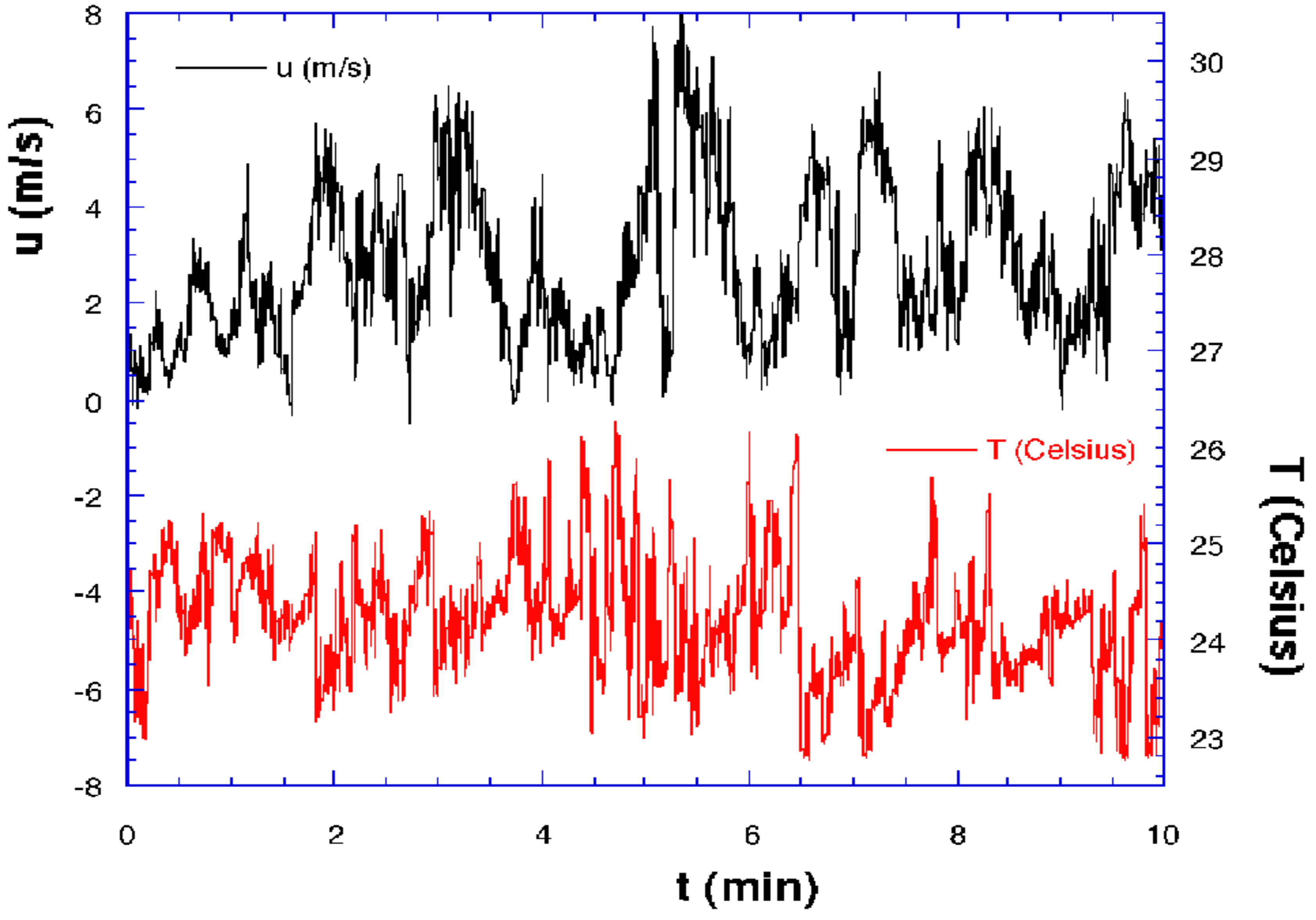
$$\begin{cases} N(A) = 0 \text{ with probability } 1 - e^{-c.m(A)} \\ \text{else with probability } e^{-c.m(A)}: \\ \qquad N(A) = \dfrac{e^{c.m(A)}}{< e^{M(A)} >} e^{M(A)} \end{cases}$$

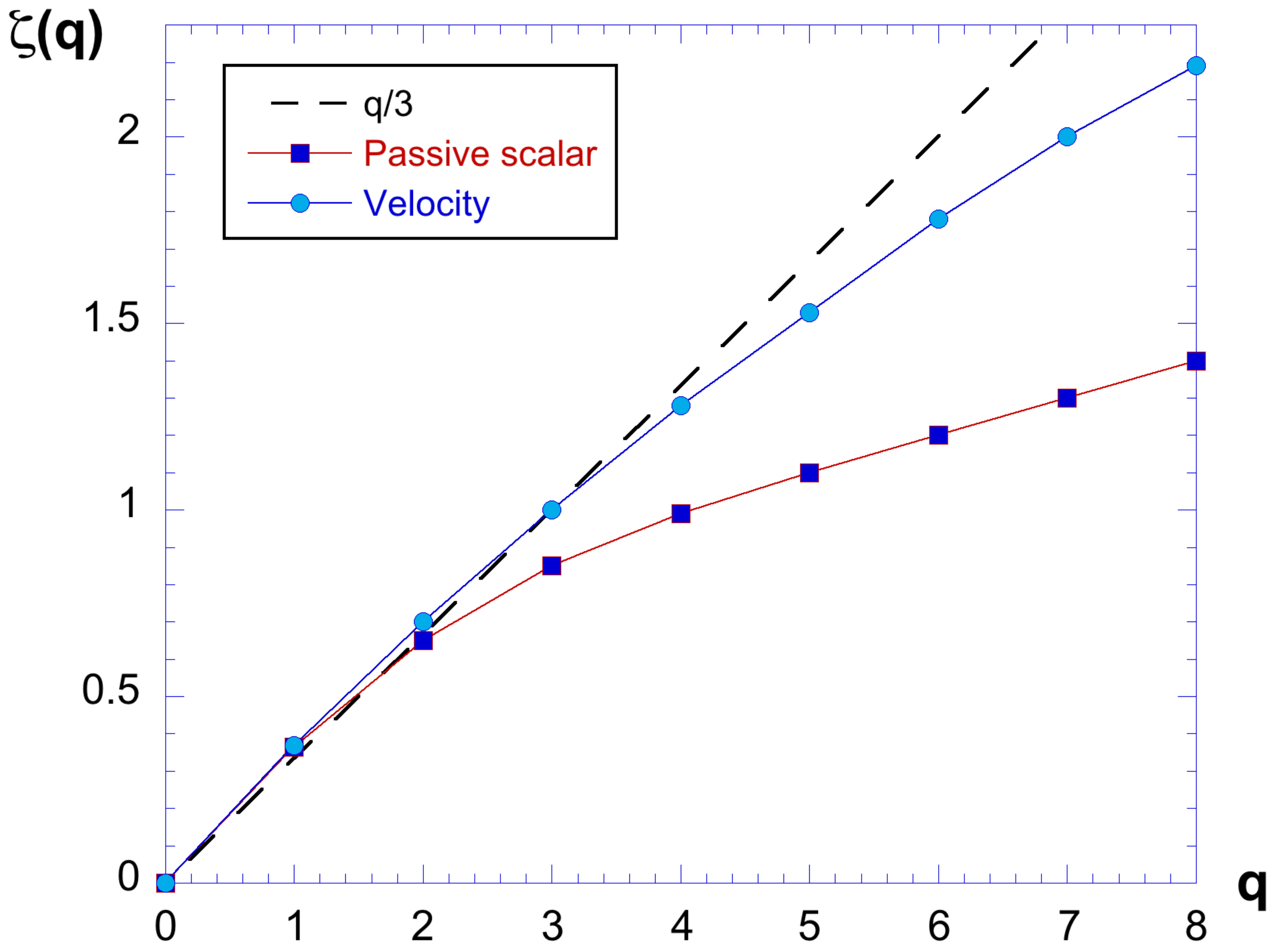

Schmitt, F.G. : Continuous multifractal models with zero values : a continuous beta-multifractal model, *Journal of Statistical Mechanics, Theory and Experiments* P02008, 2014

# Examples: non-stationary multifractal series

**Turbulence**: comparison between the velocity and passive scalars

Temperature field more intermittent than velocity field




Schmitt, F.G. and Huang Y. *Stochastic analysis of scaling times series: from turbulence theory to applications*, Cambridge University Press, 2016.

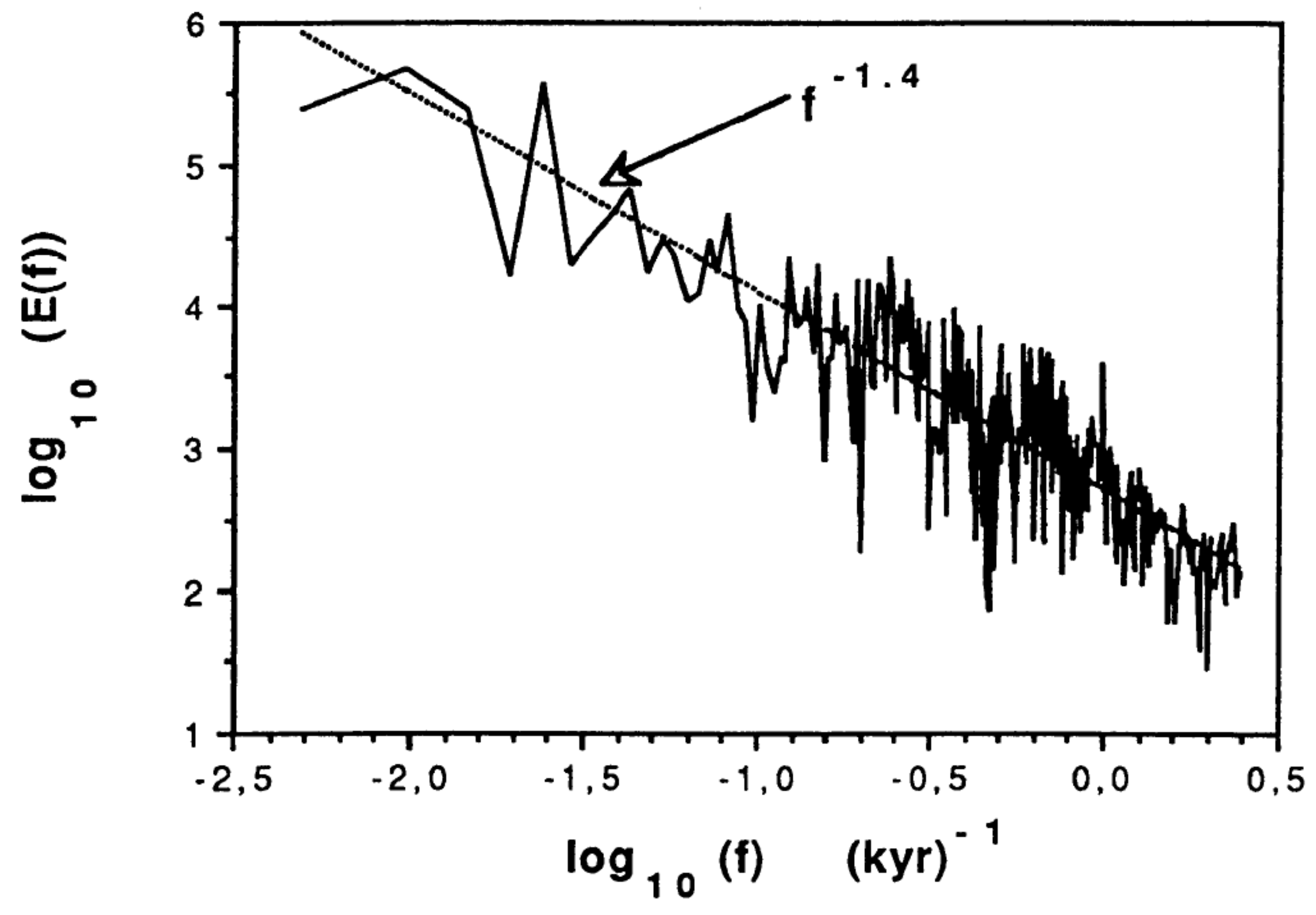
Schmitt, F.G. : *Turbulence et écologie marine*, Ellipses, Paris, 2020.

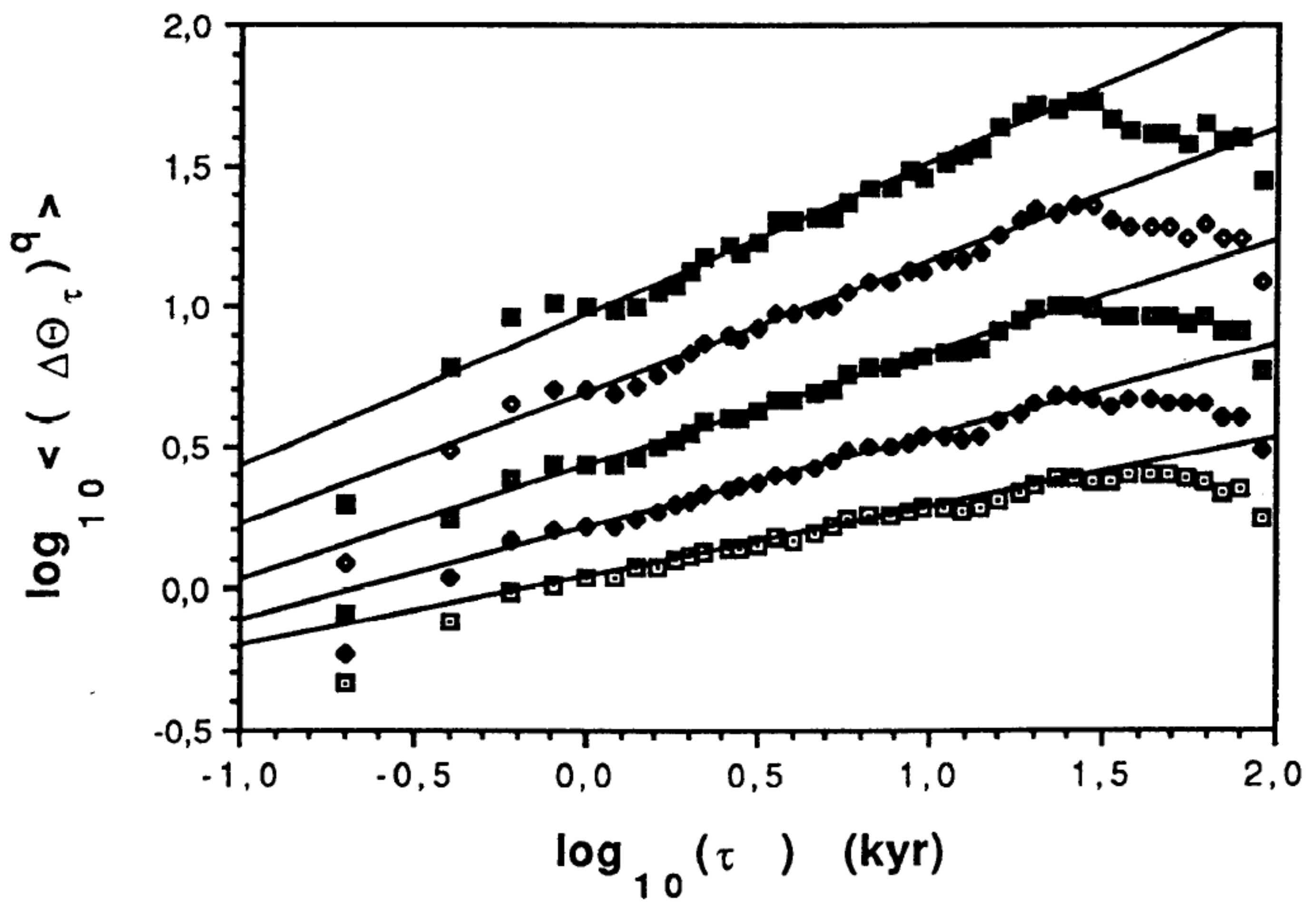
# Examples: non-stationary multifractal series

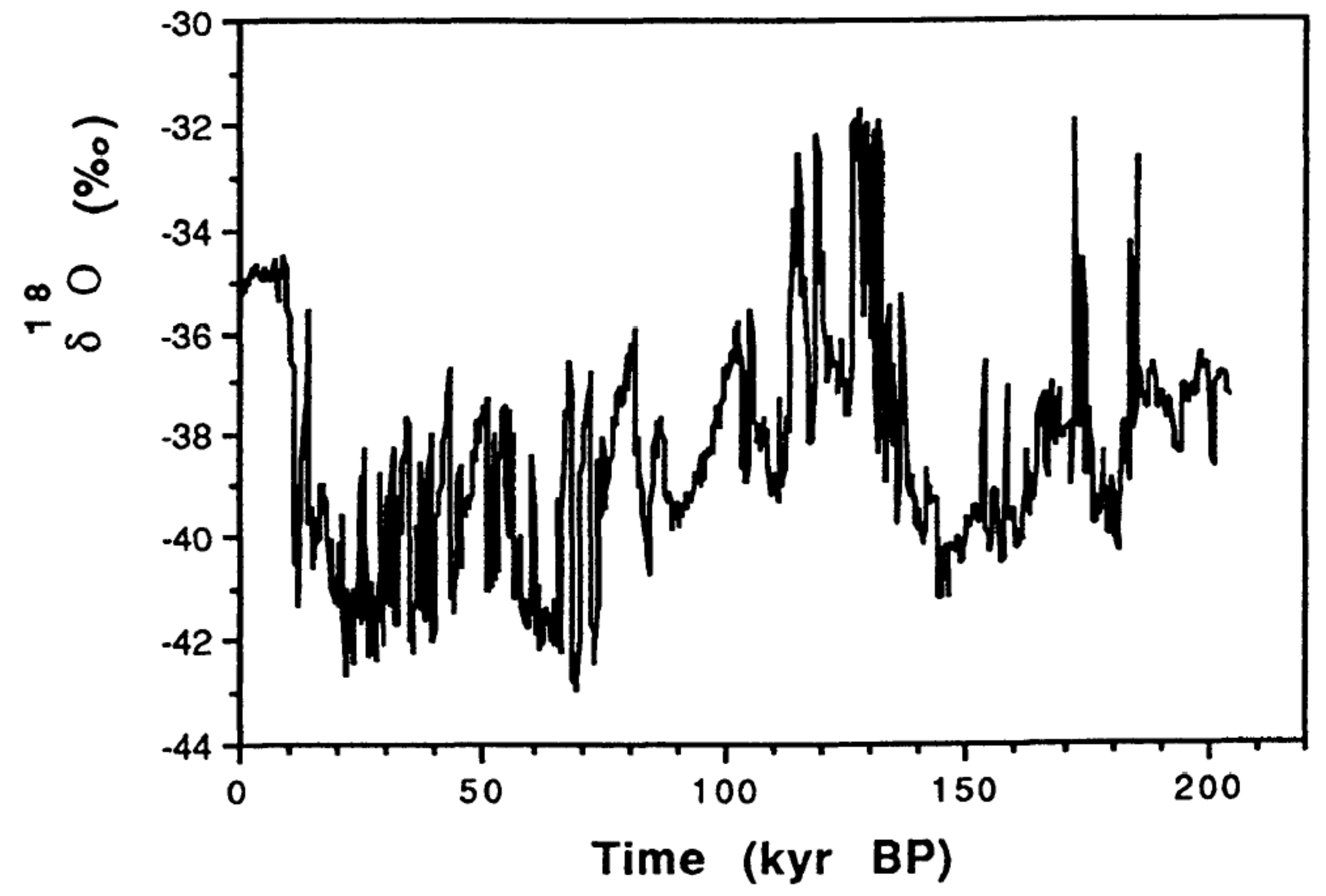
**Climate**: Greeland Ice-core

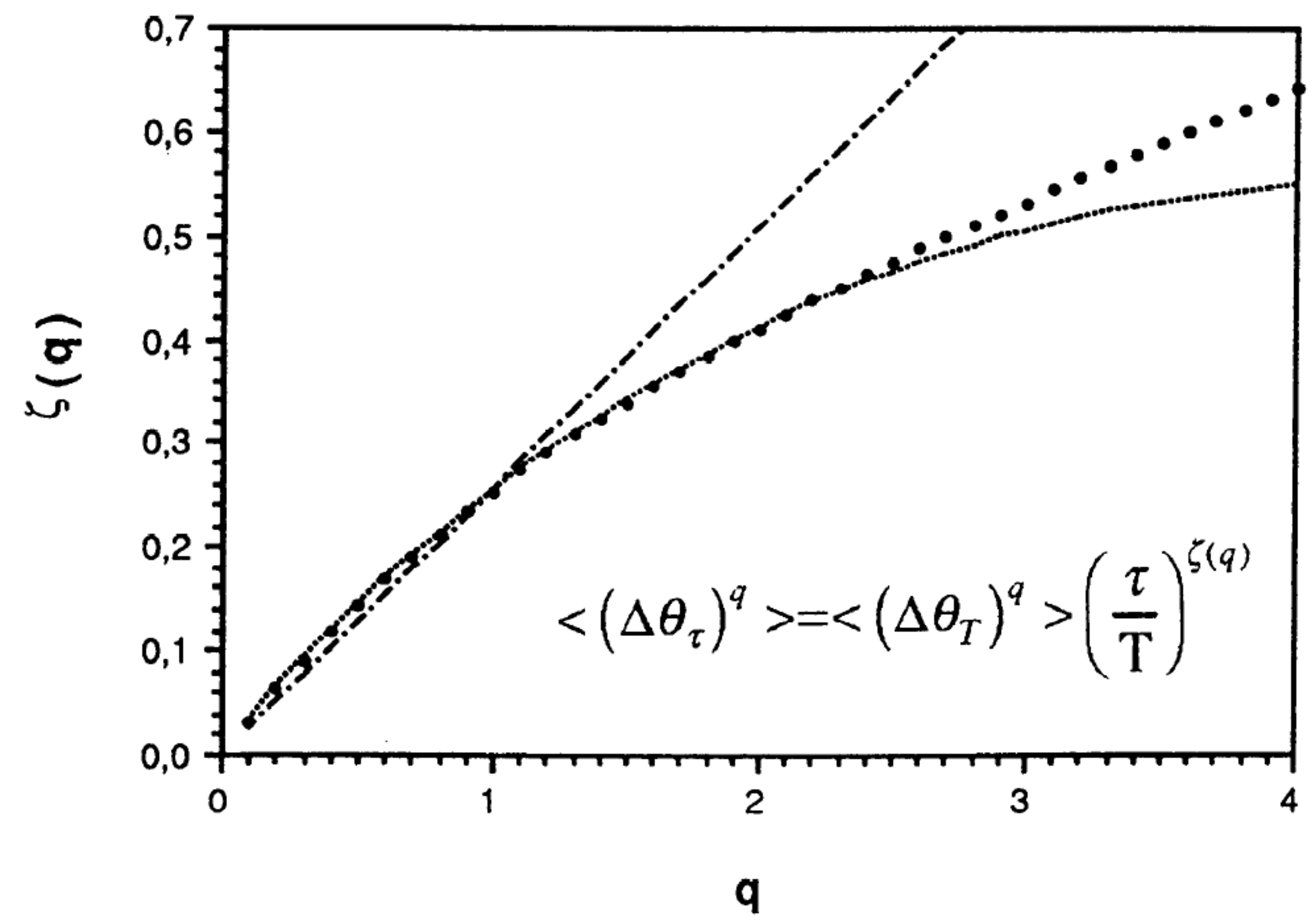
Multi-scaling from 0.4 to 40 kyr.

$H = 0.24 \pm 0.02$



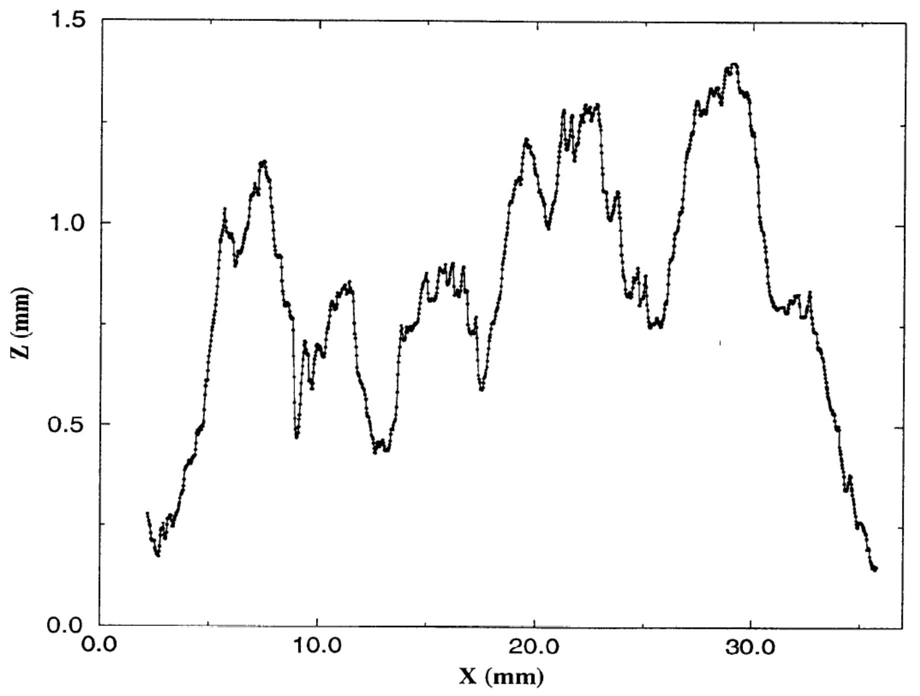



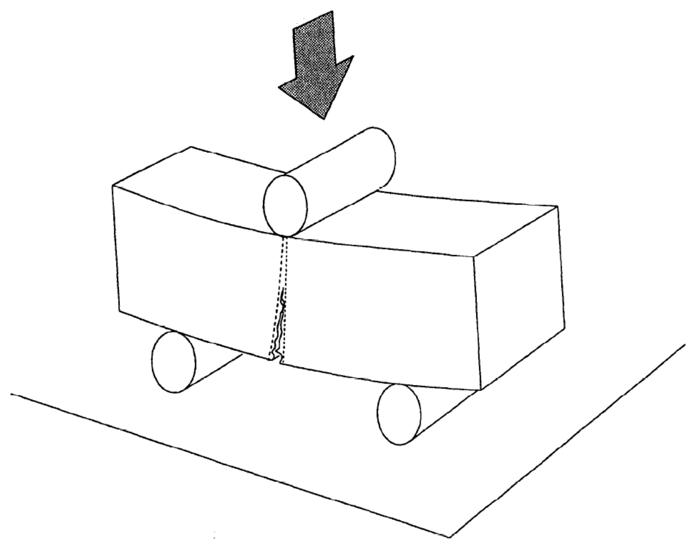



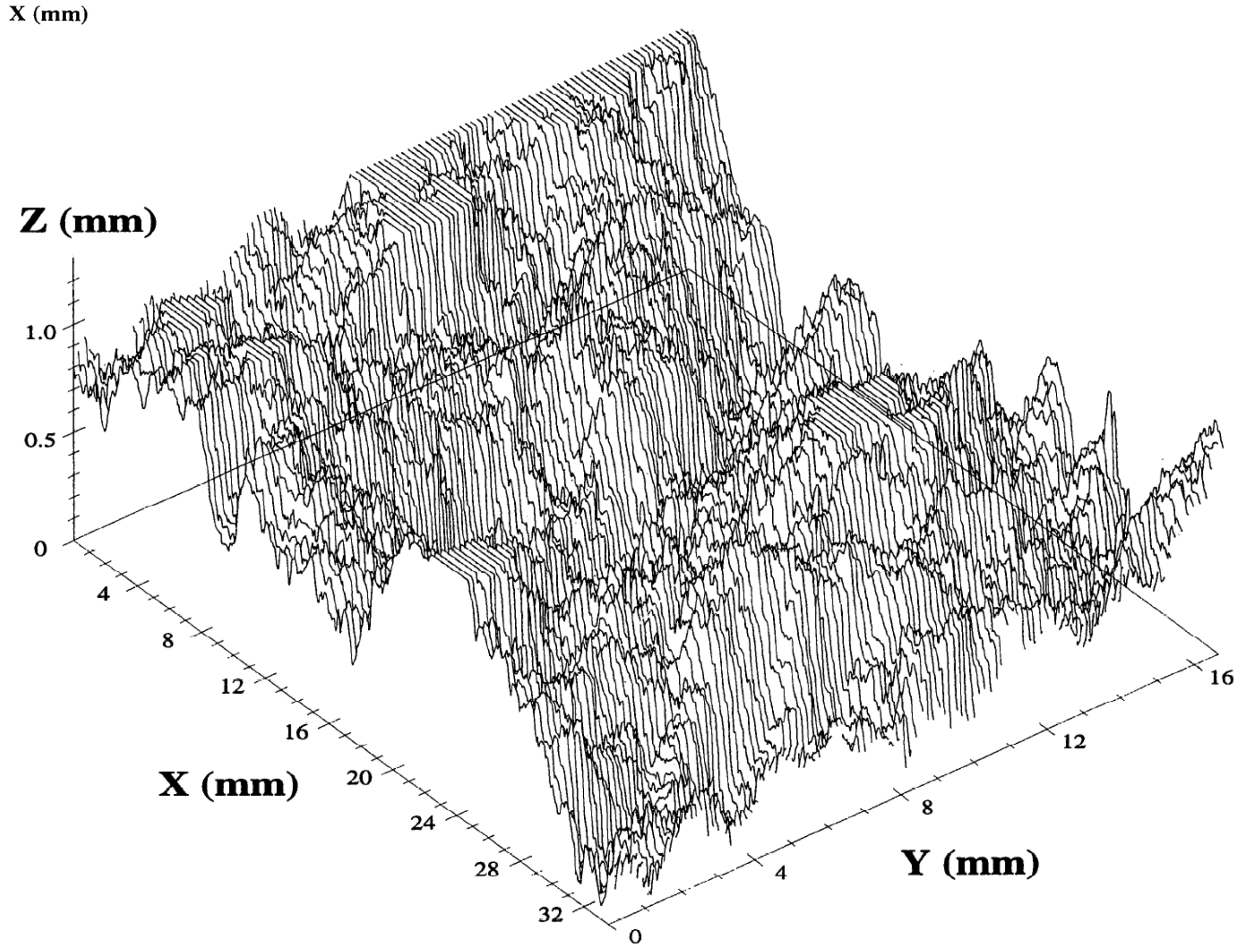

Schmitt, F., S. Lovejoy, and D. Schertzer: Multifractal analysis of the Greenland ice-core Project climate data, *Geophysical Research Letters*, 22, 1689-1692, 1995.

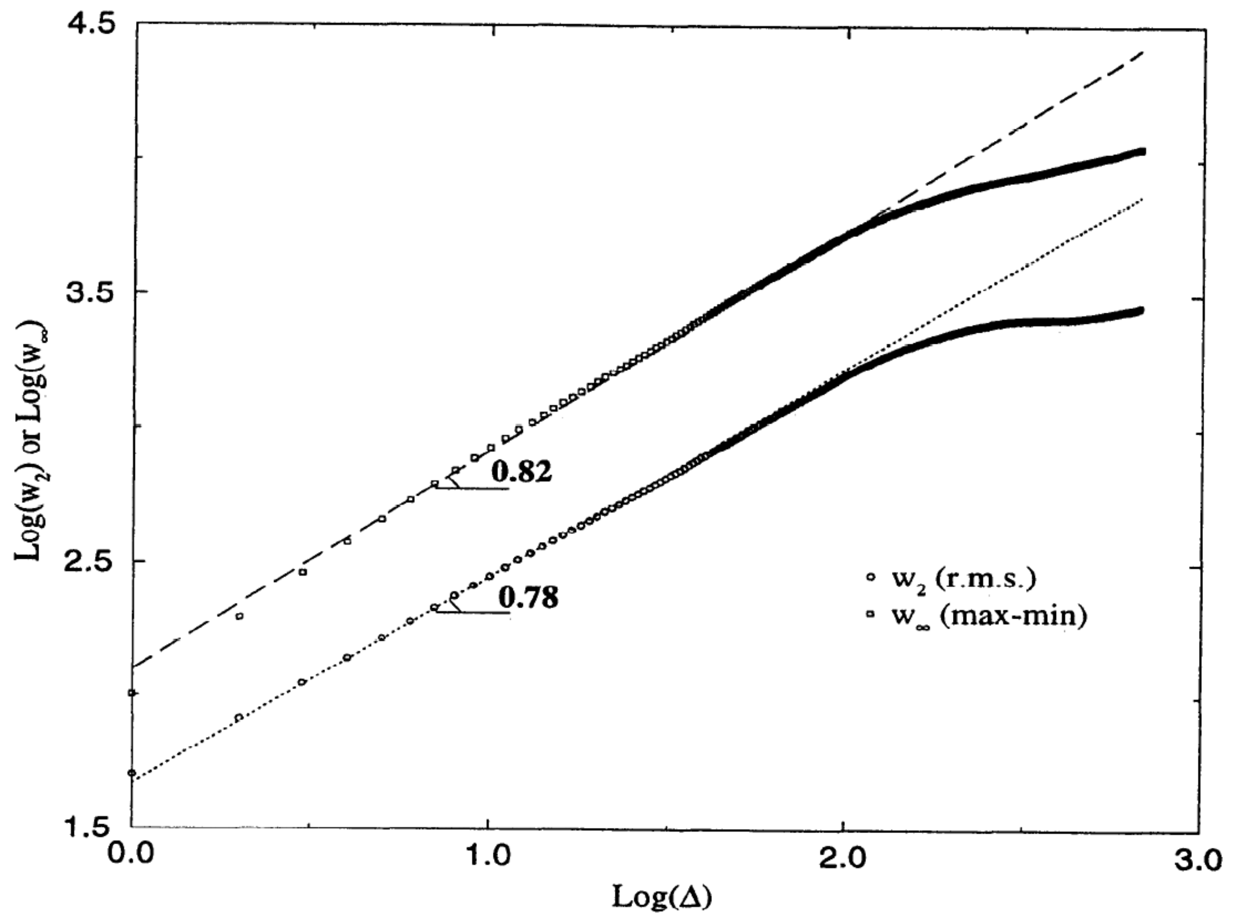
# Examples: non-stationary multifractal series

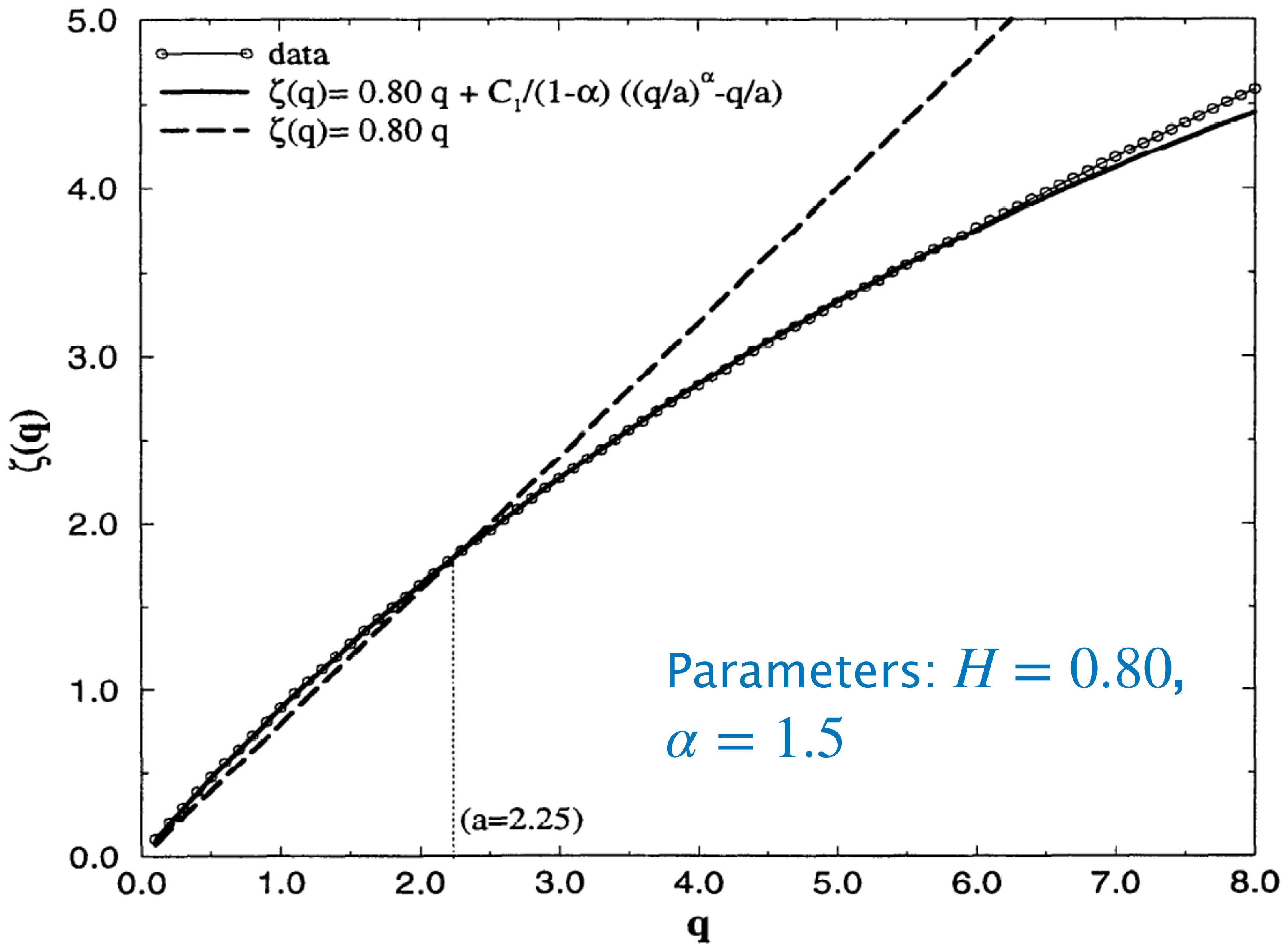
## **Fracture**: scale invariance of crack surfaces





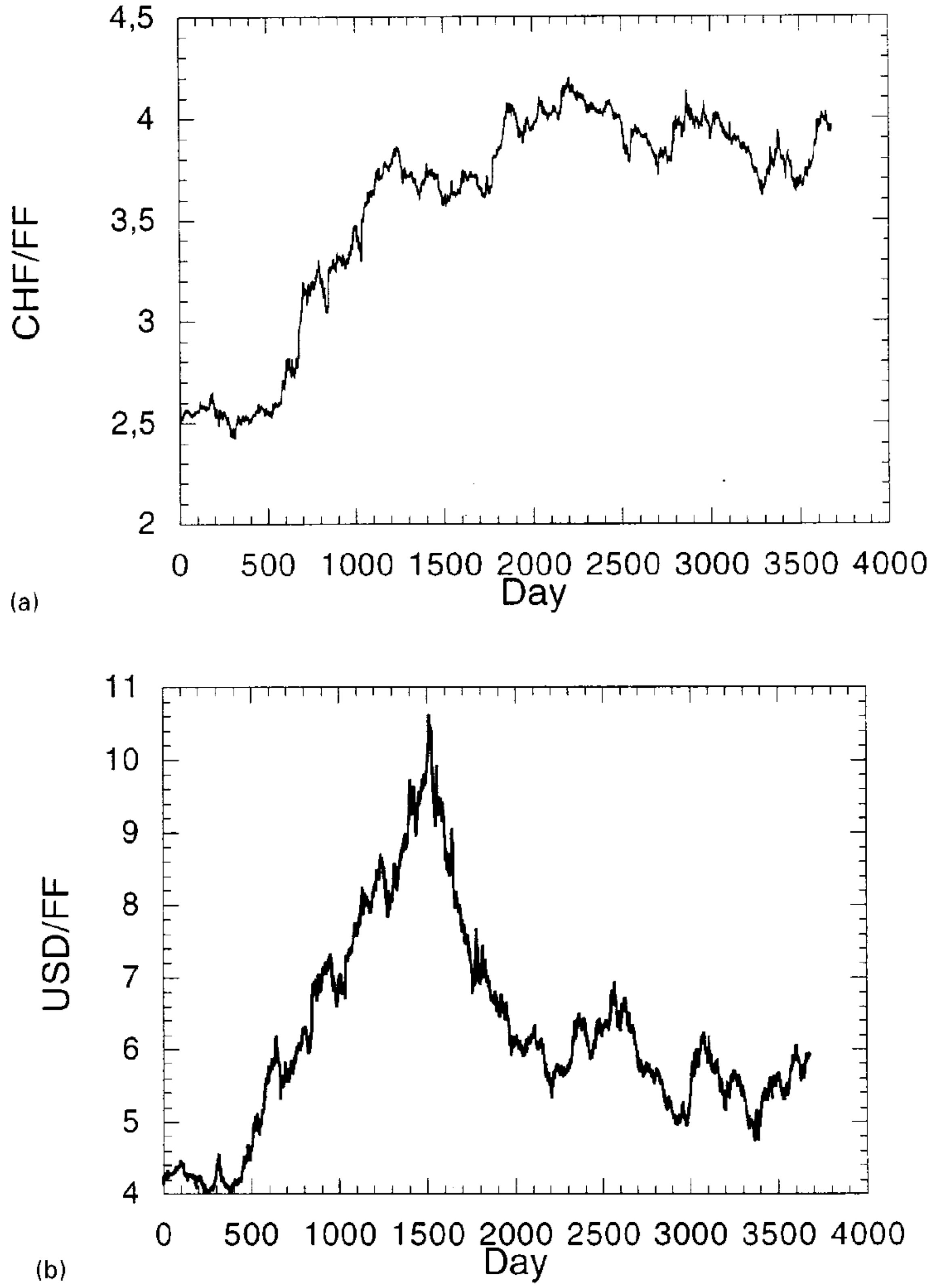



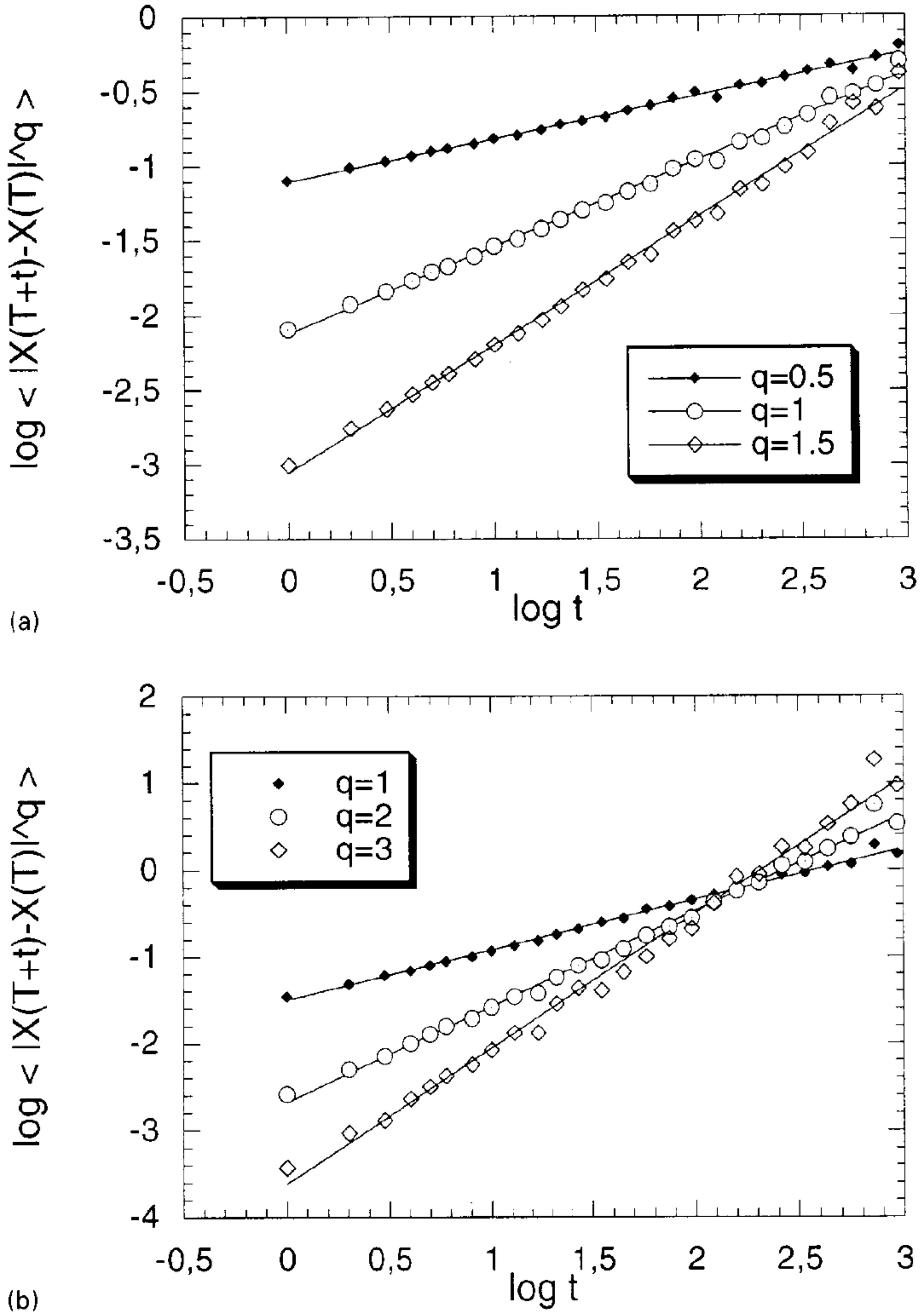

Parameters: $H = 0.80$, $\alpha = 1.5$

Schmittbuhl, J., F. Schmitt, and C. Scholz: Scaling invariance of crack surfaces, *Journal of Geophysical Research*, 100, B4, 5953-5973, 1995.

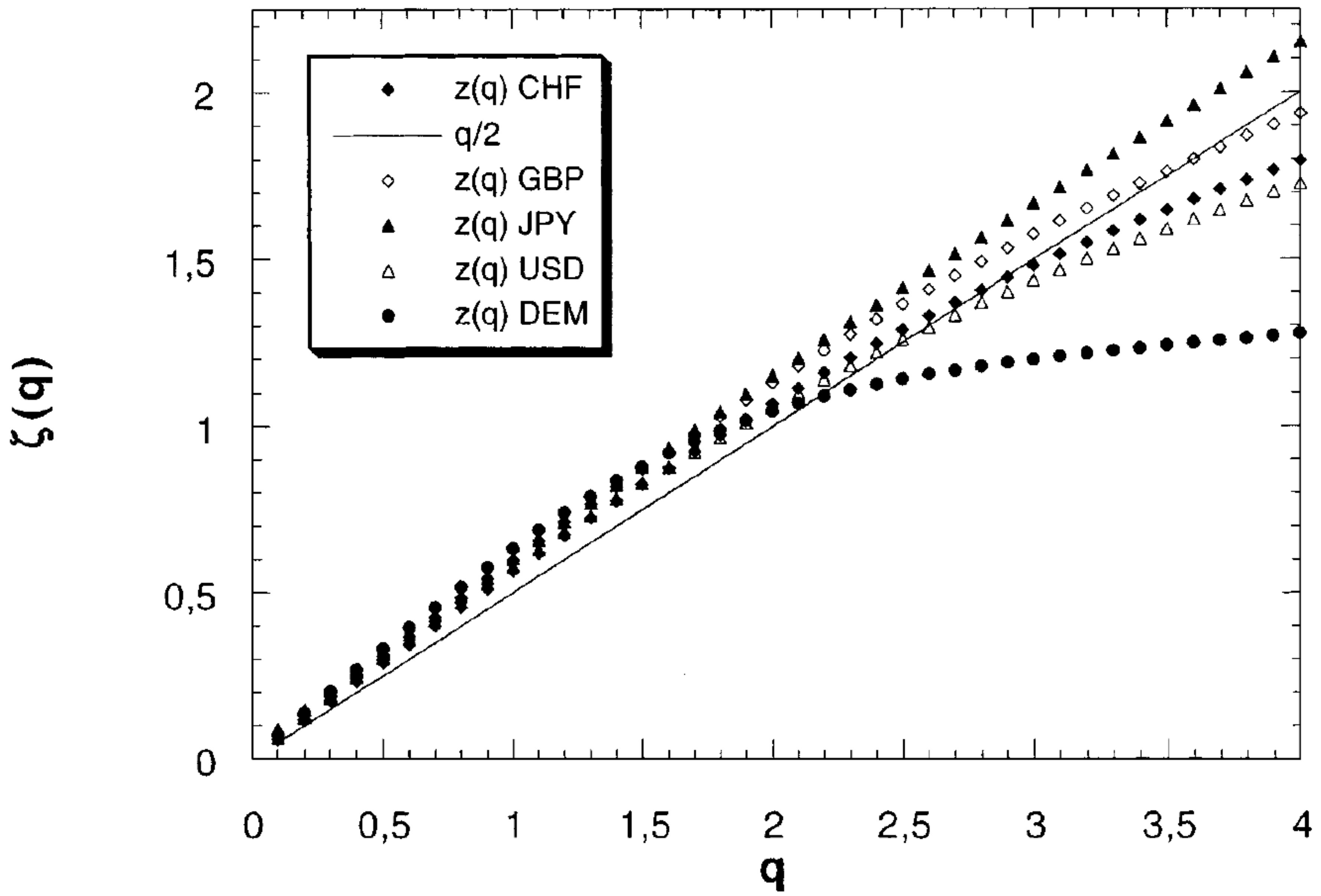
# Examples: non-stationary multifractal series

## **Finance**: exchange rates




$$\langle (\Delta Z_\tau(t))^q \rangle \sim \langle (\Delta Z_T)^q \rangle \left( \frac{\tau}{T} \right)^{\zeta(q)}$$




Schmitt, F., D. Schertzer, and S. Lovejoy: Multifractal analysis of foreign exchange data, *Applied Stochastic Models and Data Analysis*, 15, 29-53, 1999.

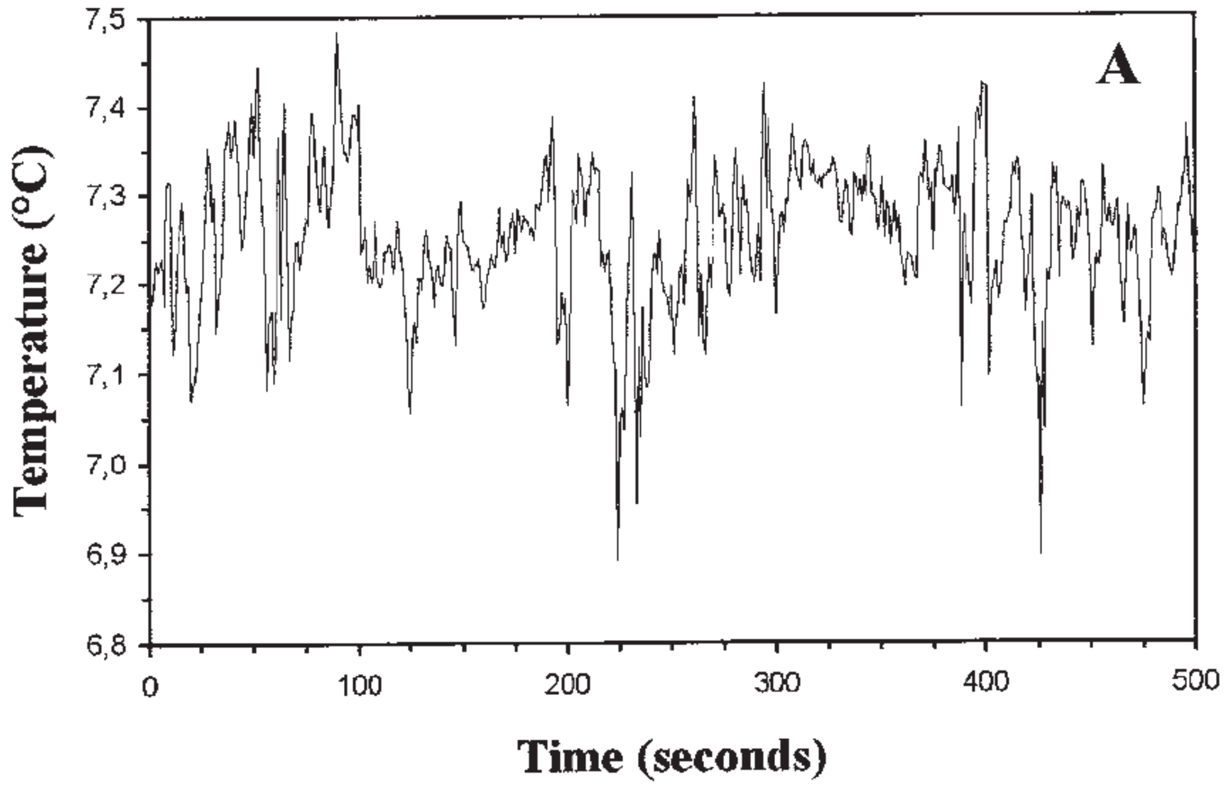
# Examples: non-stationary multifractal series

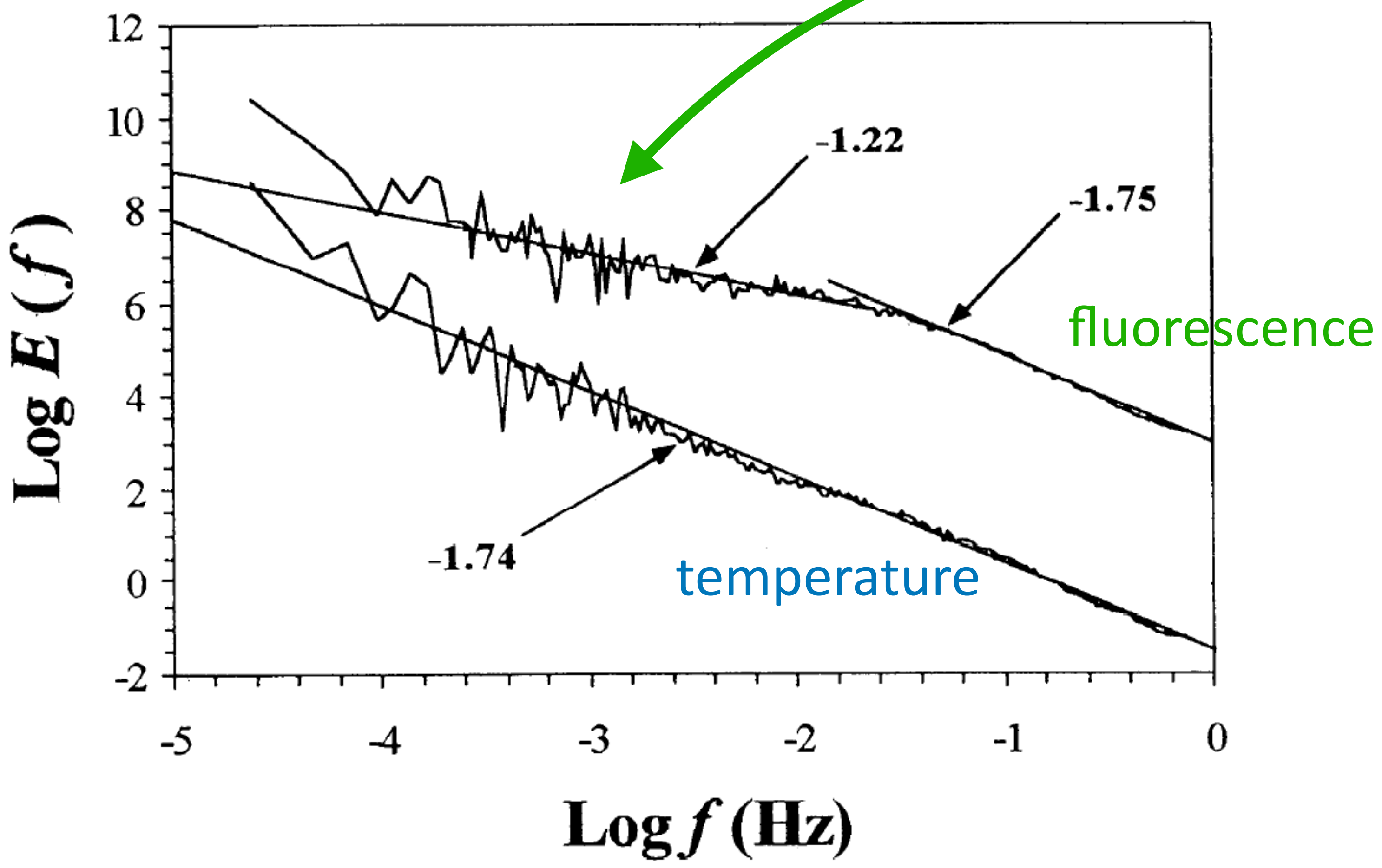
**Oceanology**: comparing temperature (passive scalar) with biologically active scalar (fluorescence, proxy of phytoplankton concentration)

$$\frac{\partial \Theta}{\partial t} + \mathbf{u}.\nabla\Theta = D\nabla^2\Theta + R(\theta)$$

transport

biological activity



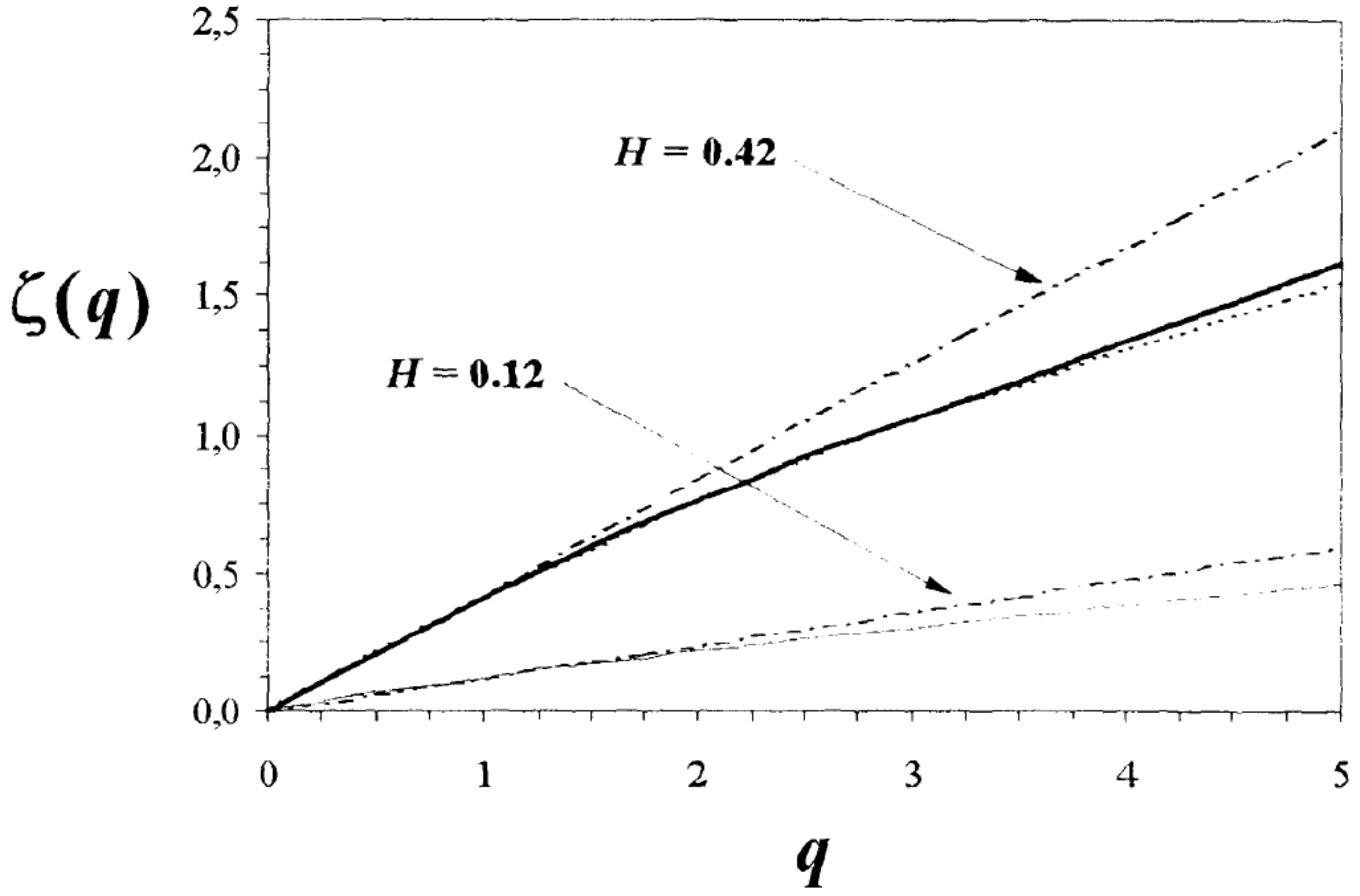




**Figure 3.** The empirical curves of scaling exponent structure functions $\zeta(q)$ for temperature (thick continuous line), small-scale (dashed line) and large-scale fluorescence (thin continuous line) compared to the theoretical monofractal linear curve $\zeta(q) = qH$ with $H = 0.42$ and $H = 0.12$ (discontinuous lines). The nonlinearity of the empirical curves indicates multifractality.


Seuront, L., F. Schmitt, Y. Lagadeuc, D. Schertzer, S. Lovejoy, and S. Frontier: Multifractal analysis of phytoplankton biomass and temperature variability in the ocean, *Geophysical Research Letters*, 23, 3591-3594, 1996.

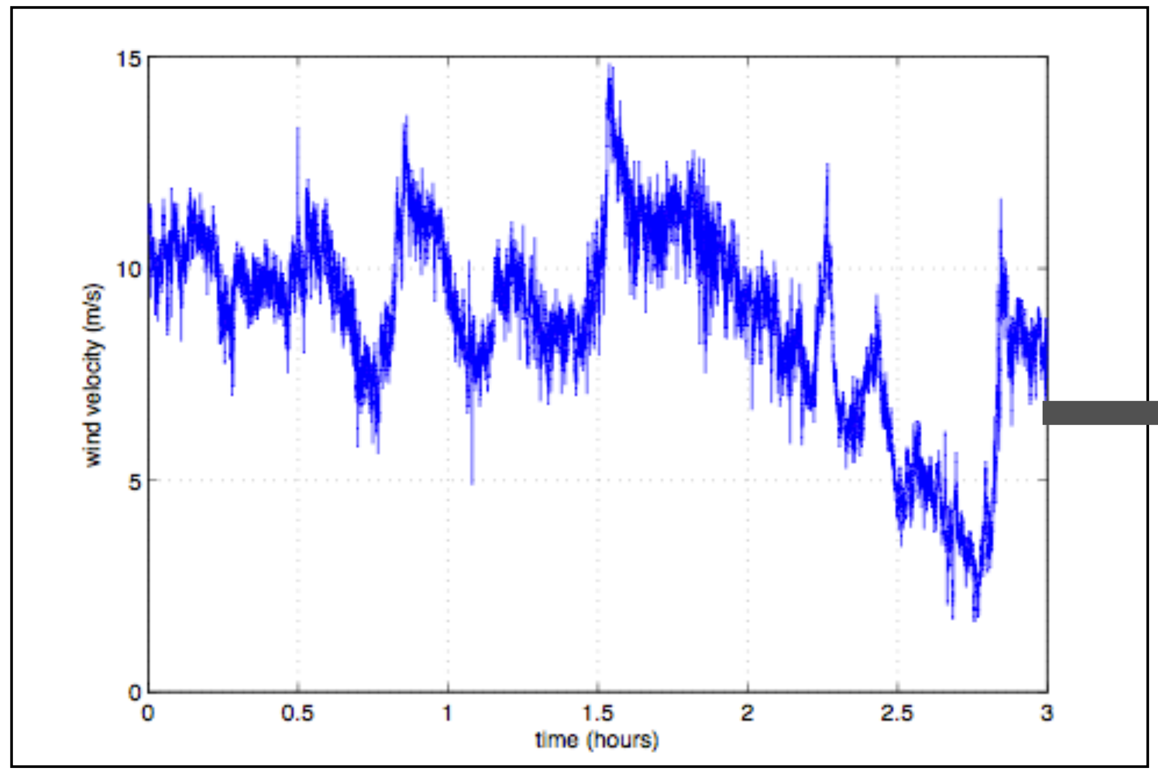
Seuront, L., F. Schmitt, D. Schertzer, Y. Lagadeuc, and S. Lovejoy: Multifractal analysis of eulerian and lagrangian variability of oceanic turbulent temperature and plankton fields, *Nonlinear Processes in Geophysics*, 3, 236-246, 1996.

Seuront L., F. Schmitt, Y. Lagadeuc, D. Schertzer, and S. Lovejoy: Universal multifractal analysis as a tool to characterise multiscale intermittent patterns; example of phytoplankton distribution in turbulent coastal waters, *Journal of Plankton Research*, 21, 877-922, 1999.

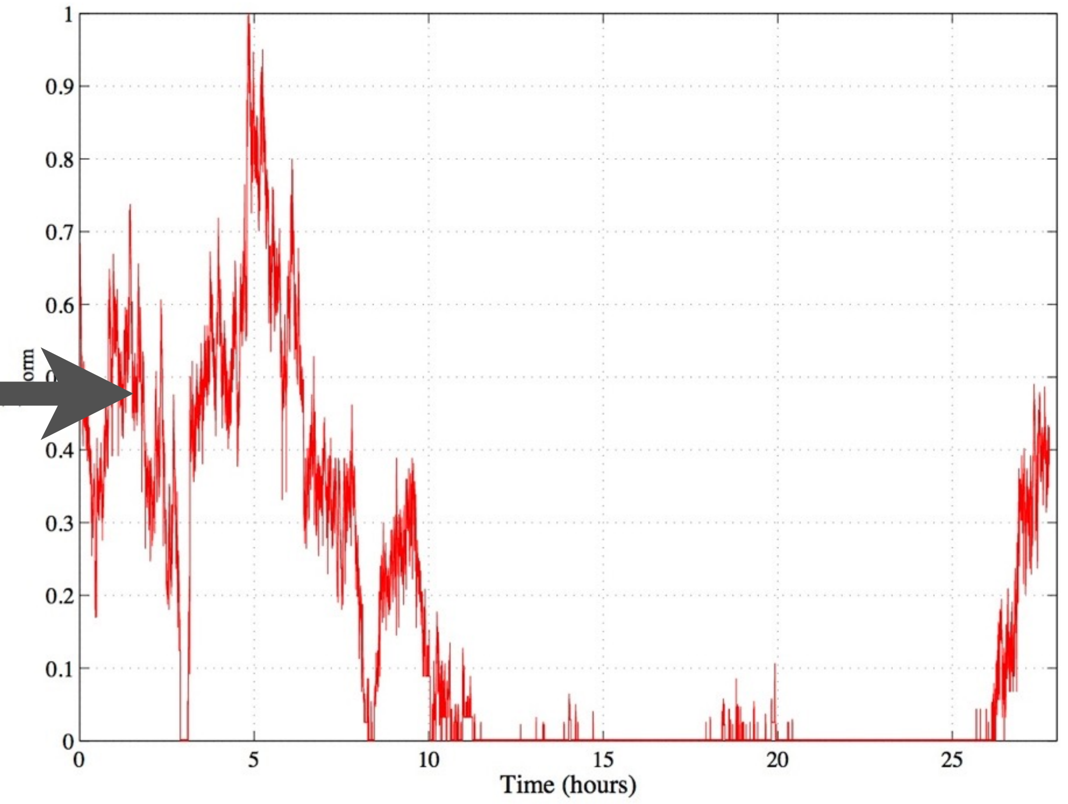
Schmitt, F.G. : Turbulence et écologie marine, Ellipses, Paris, 2020.

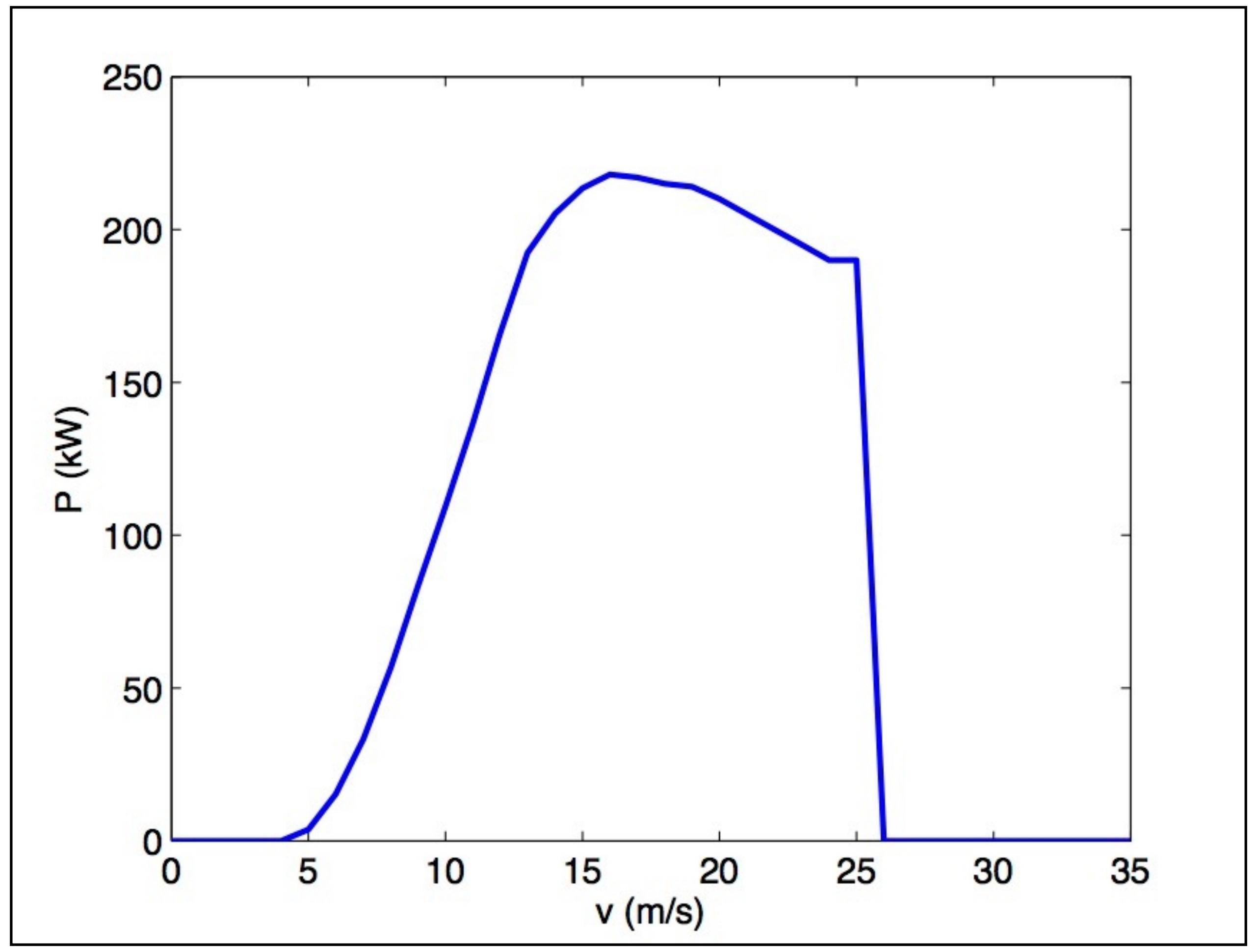
# Examples: non-stationary multifractal series

## **Wind energy**: multifractal properties of the wind power produced




wind turbine



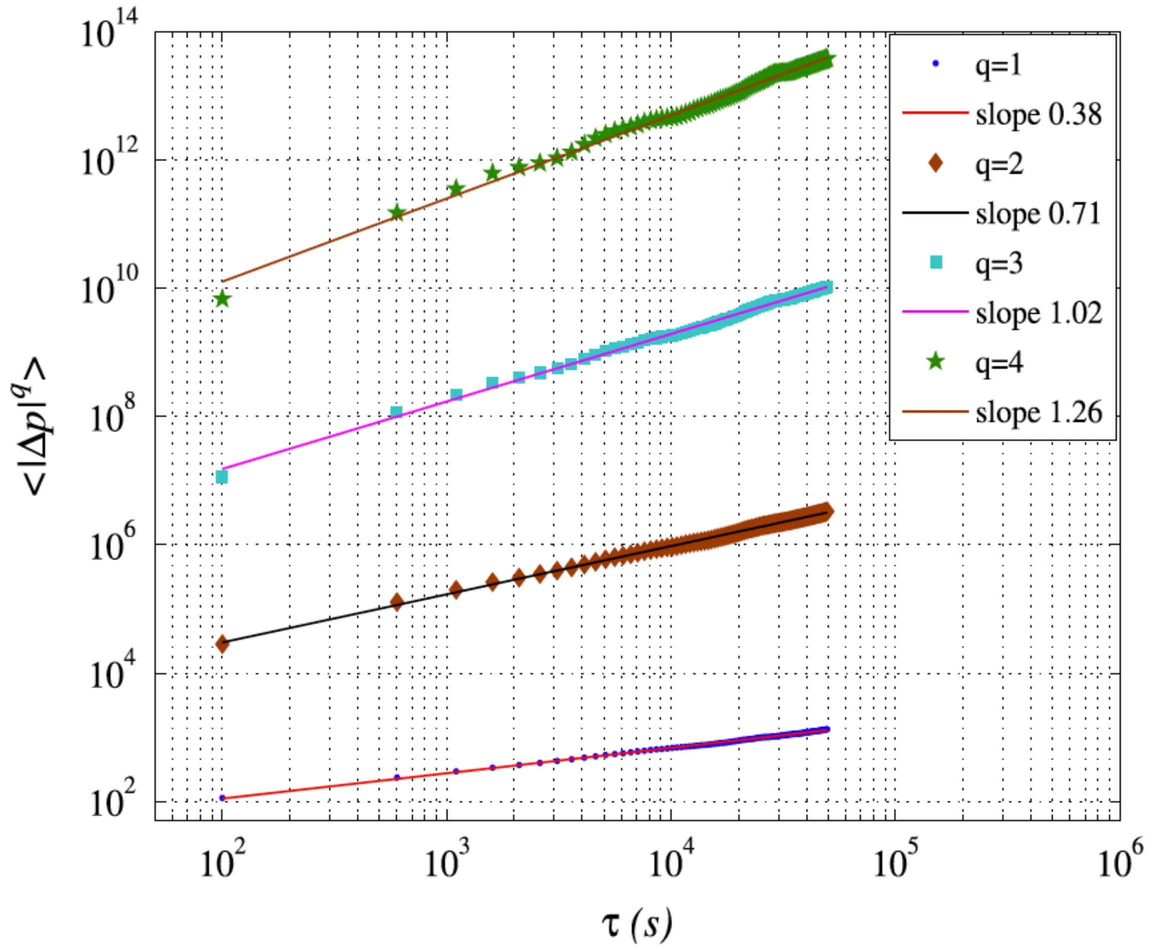




transfer function: mean power produced for a given wind input

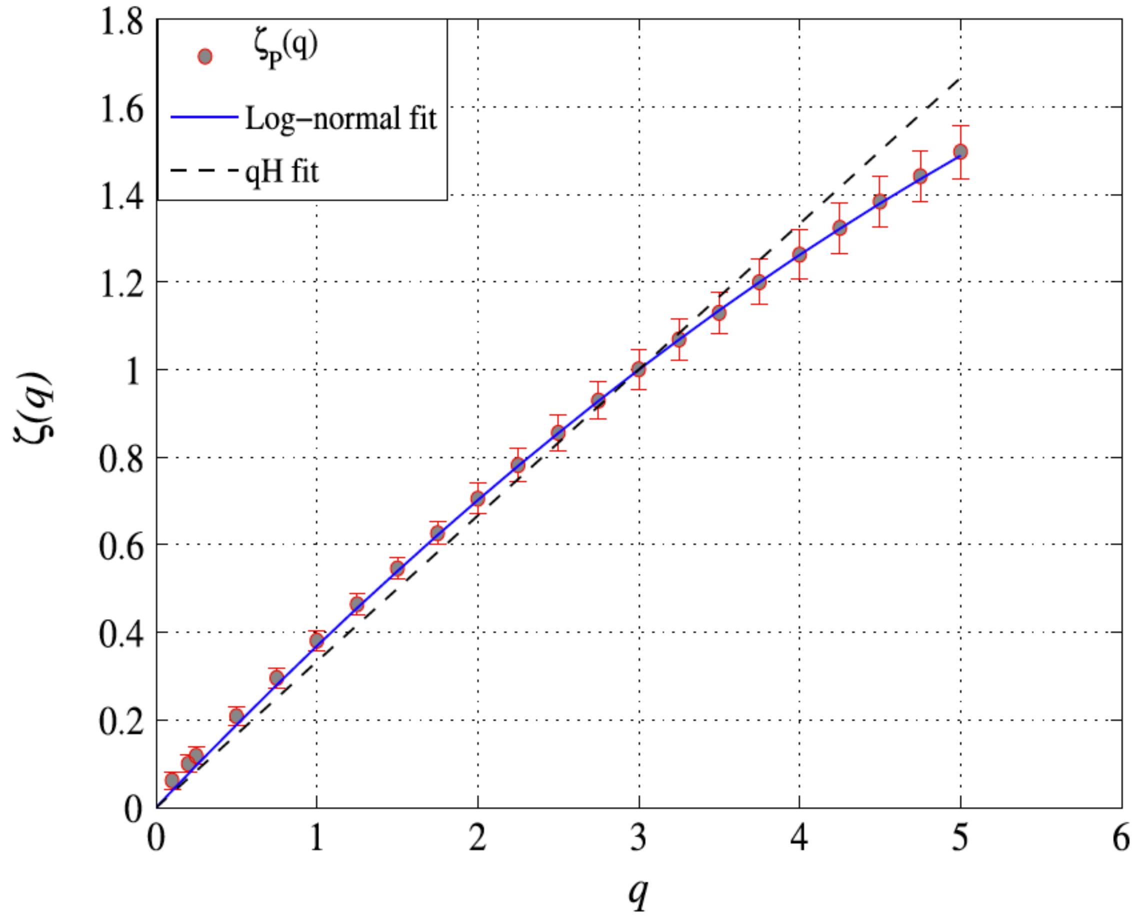

Calif, R., F.G. Schmitt, Y. Huang, Characterization of wind energy fluctuations using arbitrary-order Hilbert spectral analysis, *Physica A*, 392, 4106-4120, 2013.

Calif, R. and F.G. Schmitt, Multiscaling and joint multiscaling description of the atmospheric wind speed and the aggregate output power from a wind farm, *Nonlinear Processes in Geophysics* 21, 379-392, 2014.

Duran Medina, O., Schmitt F.G., Calif R., Multiscale analysis of wind velocity, power output and toration of a windmill, *Energy Procedia*, 76, 193-199, 2015.

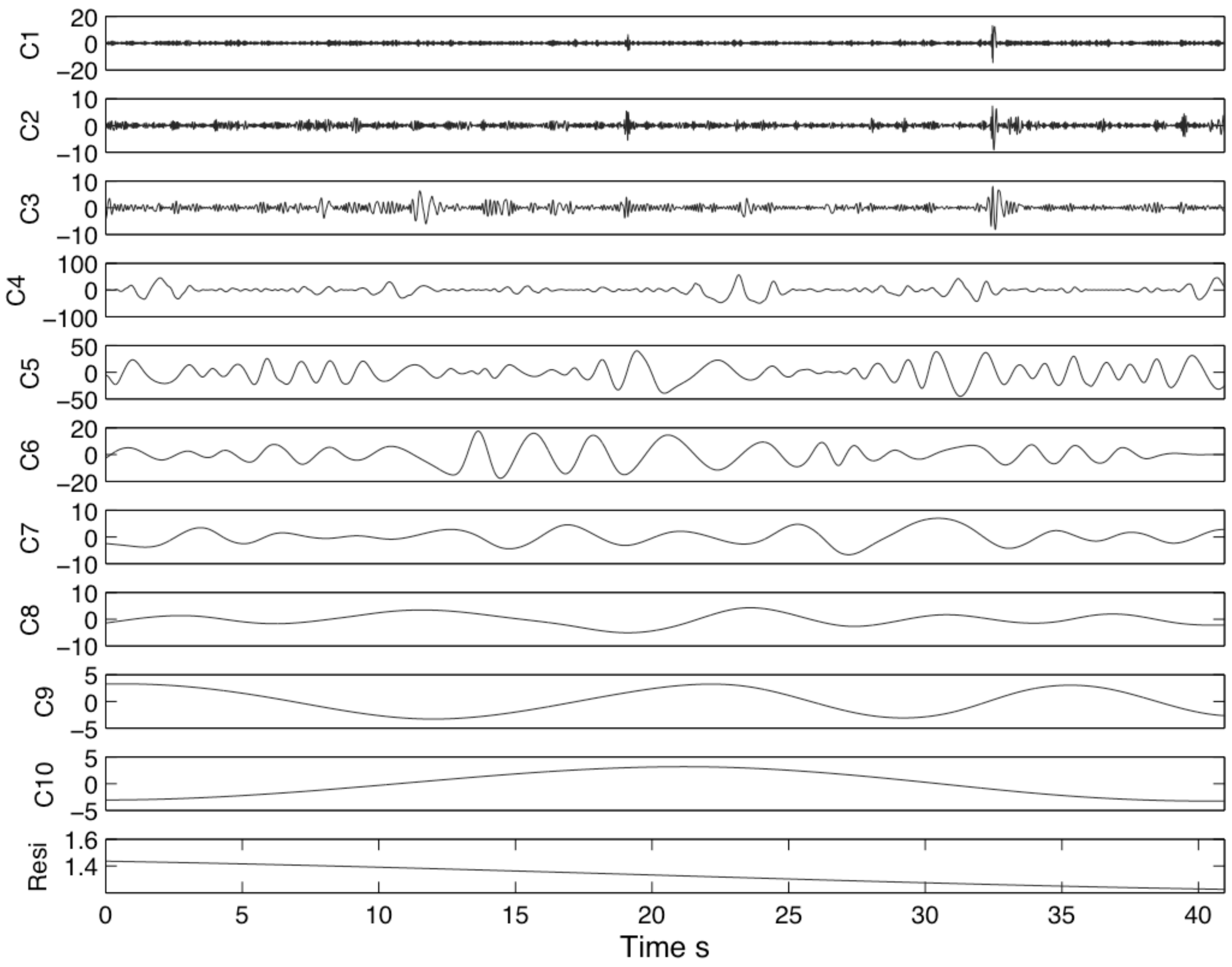

(a) Structure functions $S_q(\tau)$.




(b) Scaling exponents $\zeta(q)$.

# Another method to extract scaling exponents: EMD-HSA method

## Empirical Mode Decomposition + generalized Hilbert Spectral Analysis




$$x(t)=\sum_{i=1}^{N} C_i(t)+r_n$$

N=number of modes (order log2(number of points)

original signal

residu

Mode

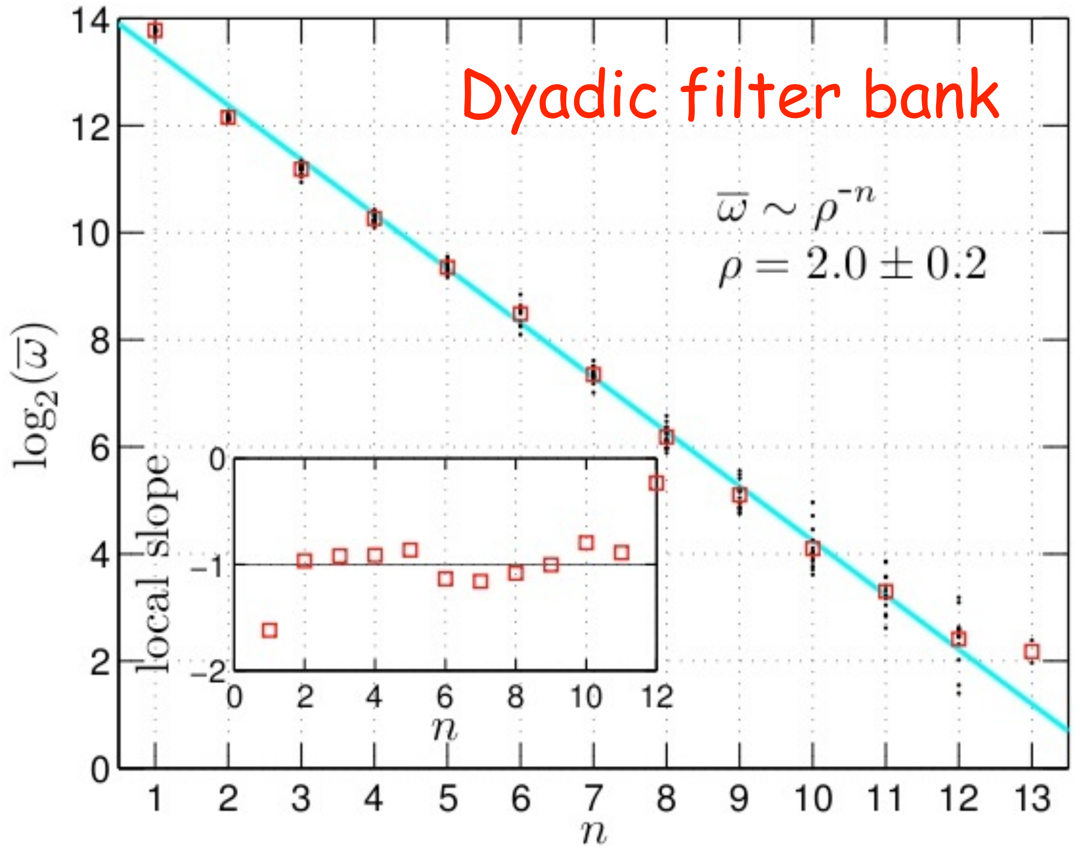
(1) Identify all local maximum (resp. minimum) extrema of x(t)

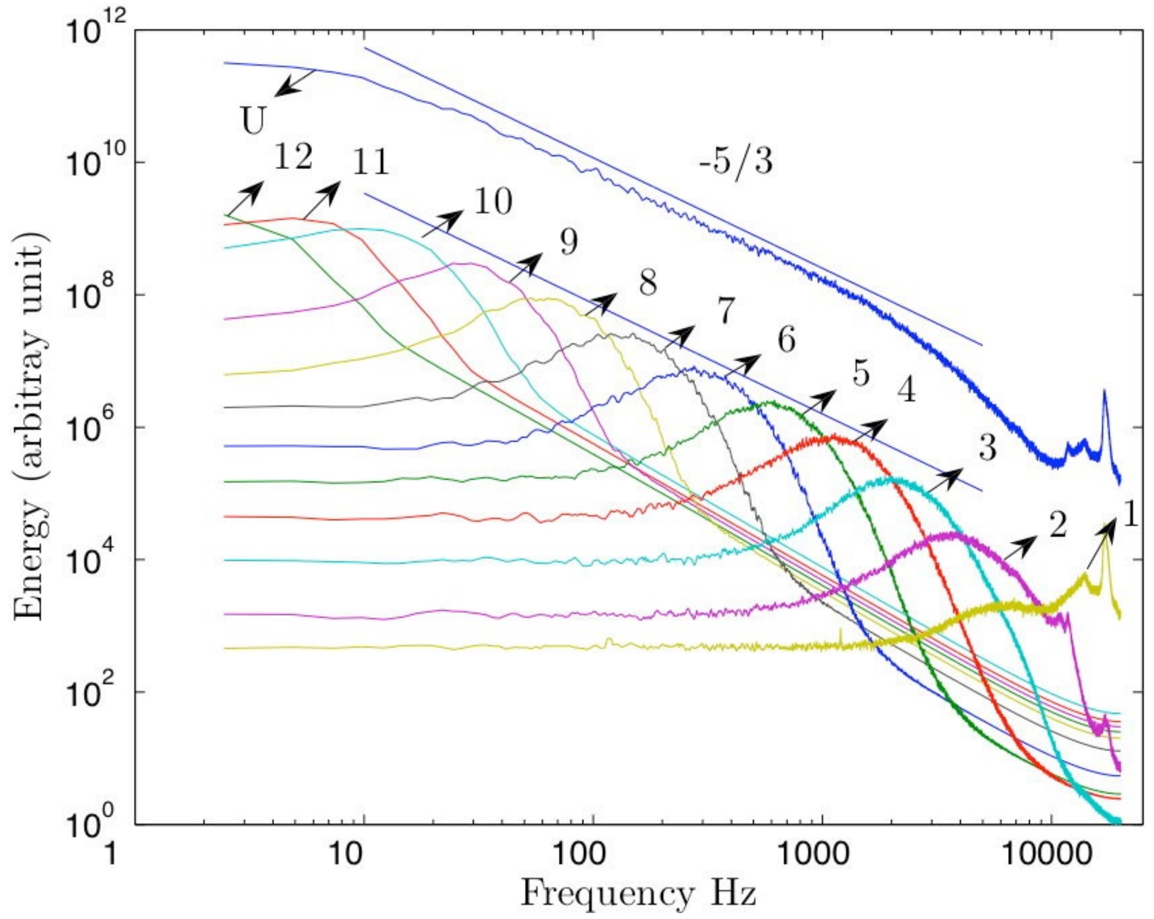
(2) Interpolate maximum (resp. minimum) by cubic spline to form upper (reps. lower) envelop: e_max and e_min

(3) Compute the average m(t)=(e_max+e_min)/2

(4) Extract the detail d(t)=X(t)-m(t)

(5) Iterate on the residual




Huang Y., F. G. Schmitt, Z. Lu, Y. Liu, An amplitude-frequency study of turbulent scaling intermittency using Hilbert spectral analysis, *EPL* 84, 40010, 2008.

Huang, Y., F.G. Schmitt, J.-P. Hermand, Y. Gagne, Z. M. Lu, Y.L. Liu, Arbitrary order Hilbert spectral analysis for time series possessing scaling statistics: a comparison study with detrended fluctuation analysis and wavelet leaders, *Physical Review E* 84, 016208, 2011.

Schmitt, F.G. and Huang Y. *Stochastic analysis of scaling times series: from turbulence theory to applications*, Cambridge University Press, 2016.

# Another method to extract scaling exponents: EMD-HSA method

## Empirical Mode Decomposition + arbitrary order Hilbert Spectral Analysis

After decomposition the signal is written $X(t) = \sum_{i=1}^{n} C_i(t) + r_n(t)$

Hilbert transform of each mode $C_i^H(t) = \int_{-\infty}^{+\infty} \frac{C_i(u)}{t-u} du$

Construction of an "analytical" signal $C_i^A(t) = C_i(t) + jC_i^H(t) = A_i(t)e^{j\theta_i(t)}$

At each time step, extraction of a local amplitude *A*(t) and local frequency $\omega_i(t) = \frac{d}{dt}\theta_i(t)$

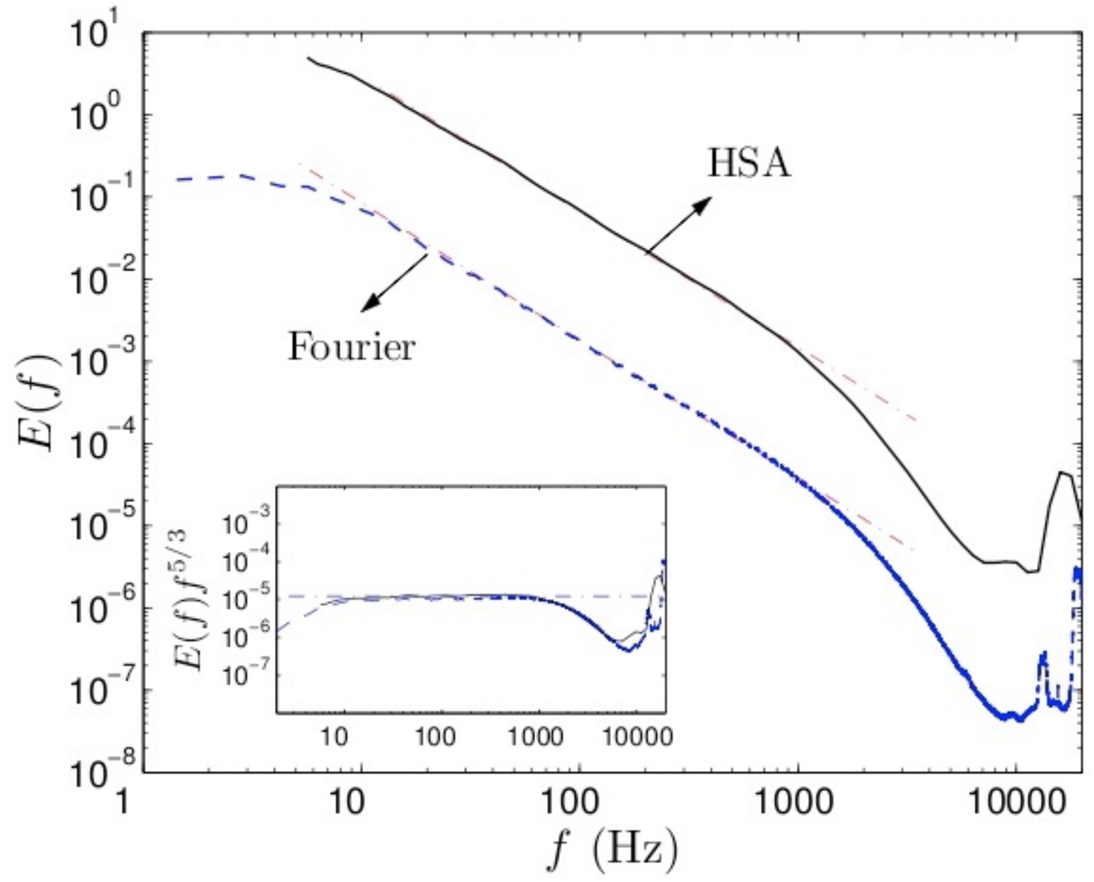
**A time-frequency-amplitude analysis**

Joint probability density function (frequency and amplitude) $p(\omega, A)$

Estimation of energy $h(\omega) = \int_0^\infty A^2 p(\omega, A) dA$

**This is called Hilbert Spectral Analysis**

Example, turbulence 5/3 spectrum:

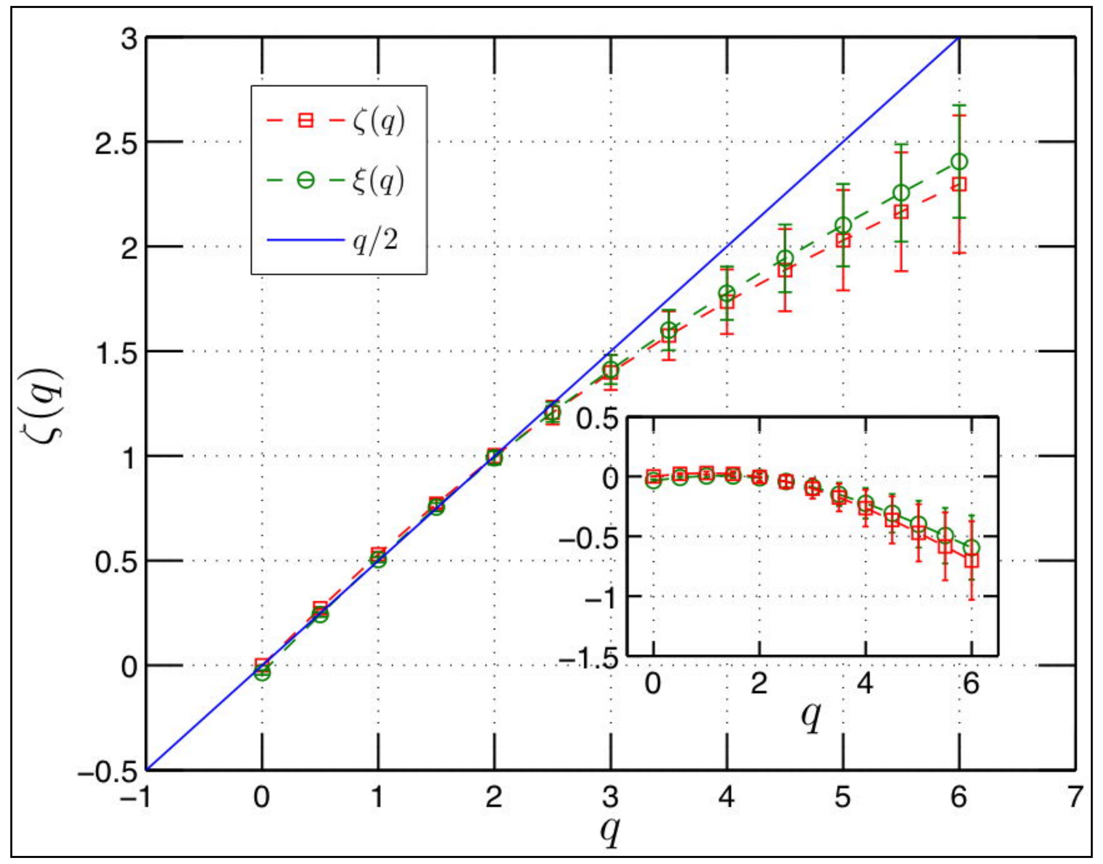



**Our contribution: arbitrary order HSA**

$$L_q(\omega) = \int_0^\infty A^q p(\omega, A) dA$$

$$L_q(\omega) \sim \omega^{-\xi(q)}$$

$$\xi(q) = 1 + \zeta(q)$$




Estimation of the multifractal moment function in the spectral space.
Compares nicely with other methods (SF, wavelet leaders, DFA).
Less influenced by periodicities in the series.


Huang Y., F. G. Schmitt, Z. Lu, Y. Liu, An amplitude-frequency study of turbulent scaling intermittency using Hilbert spectral analysis, *EPL* 84, 40010, 2008.

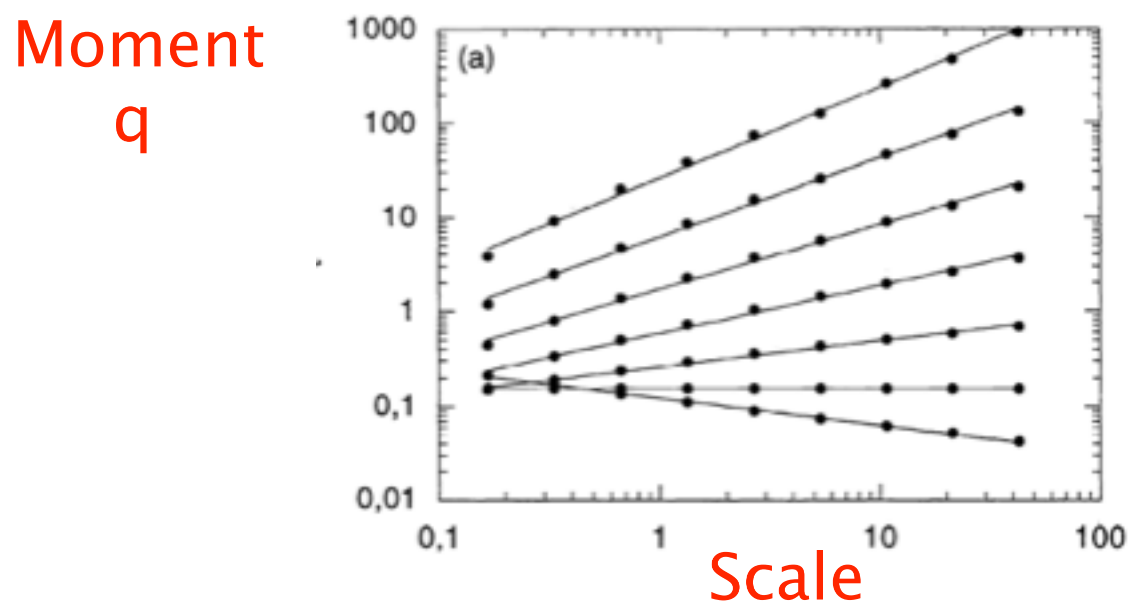
Huang, Y., F.G. Schmitt, J.-P. Hermand, Y. Gagne, Z. M. Lu, Y.L. Liu, Arbitrary order Hilbert spectral analysis for time series possessing scaling statistics: a comparison study with detrended fluctuation analysis and wavelet leaders, *Physical Review E* 84, 016208, 2011.

Schmitt, F.G. and Huang Y. *Stochastic analysis of scaling times series: from turbulence theory to applications*, Cambridge University Press, 2016.

# Use of cumulants

**Classical scaling methods**:

- Estimate moments at different scale resolutions
- Display the scale invariance of the moments
- Estimate moments functions $\zeta(q)$ as slope of a fixed moment order, over a scale range
- Extract the parameters of a given multifractal model from the moment function $\zeta(q)$




**Cumulant approach**:

- Estimate the cumulant generating function at a given scale $\Psi(q) = \log < |V(x+\ell) - V(x)|^q >$
- Extract the parameters of the multifractal model at this scale
- Change the scale, and display the scale-dependence of the parameters

**Advantages**: a better precision for a given scale because the cumulant generating function is precisely estimated; can be used even when the scale invariance is not well verified: **intermittency without perfect scaling**

Delour et al., 2001; Eggers et al., 2001; Chevillard et al., 2005; Venugopal et al., 2006.

Schmitt FG: Experimental analysis of cumulants scaling properties in fully developed intermittent turbulence, in *Nonlinear Science and Complexity*, edited by A. C. J. Luo, L. Dai and H. R. Hamidzadeh, World Scientific, 2007, pp. 240-246.

Schmitt FG, Y Huang, Z. Lu, Y. Liu, N. Fernandez, Analysis of turbulent fluctuations and their intermittency properties in the surf zone using empirical mode decomposition, *Journal of Marine Systems* 77, 473-481, 2009.

Michalec F.G., Schmitt F.G., Souissi S., Holzner M., Characterization of intermittency in zooplankton behaviour in turbulence, *European Physical Journal* E 38, 108, 2015.

# Use of cumulants

Cumulant generating function of $g_\ell = \log|V(x+\ell) - V(x)|$:
$\Psi(q) = \log < \exp(qg_\ell) > = \log < |V(x+\ell) - V(x)|^q >$

This function is convex, as a second characteristic function, and can be developed using the cumulants:

$$\Psi(q) = C_1 q + \frac{1}{2!}q^2 C_2 + \frac{1}{3!}q^3 C_3 + \ldots = \sum_{p=1}^{+\infty} \frac{q^p}{p!} C_p$$

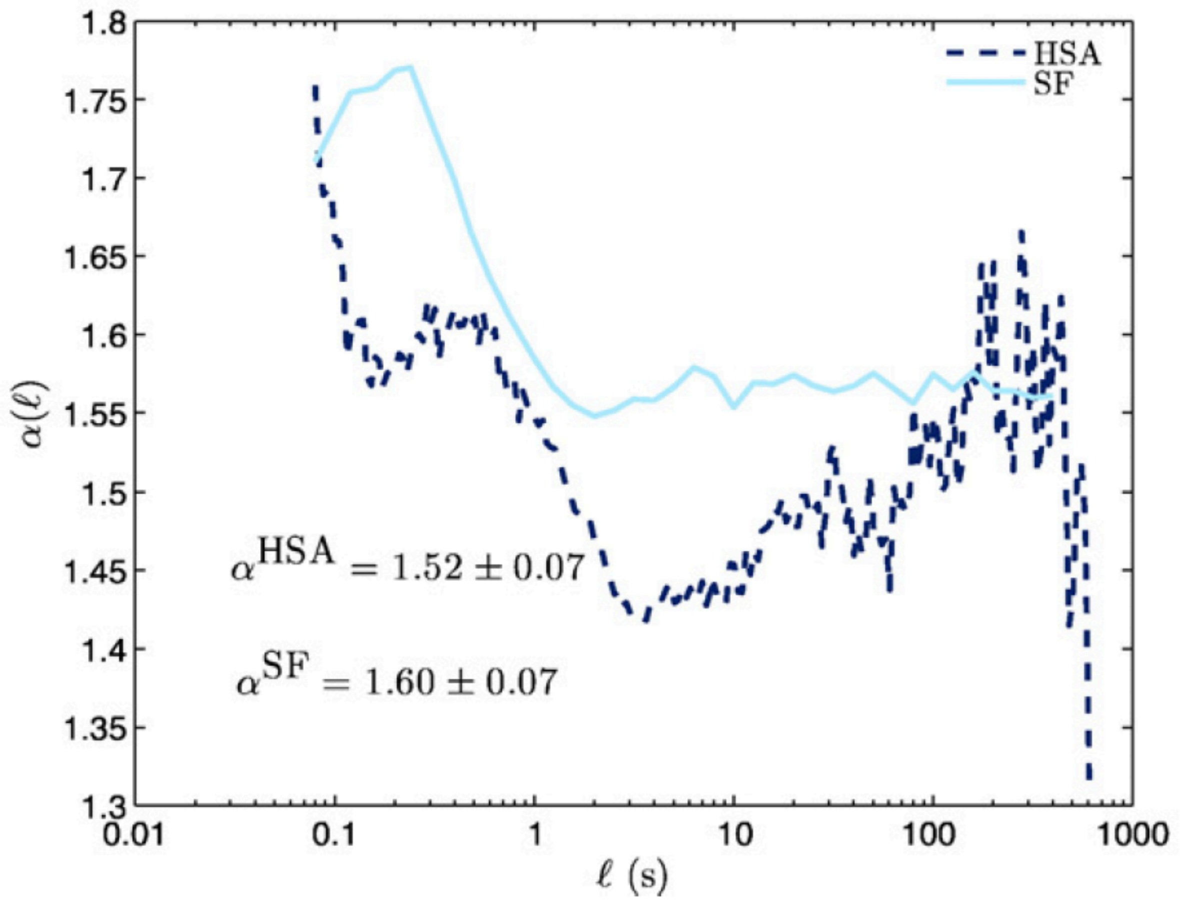
The first cumulant is $C_1 = < g_\ell > = < \log|V(x+\ell) - V(x)| >$

For the log-stable model, the development is non-analytical and we have:

$$\Psi(q) = C_1 q + C_\alpha q^\alpha$$

At scale $\ell$, $\alpha$ and $C_\alpha$ can be estimated precisely by plotting in log-log plot $\Psi(q) - C_1 q$ versus $q$.

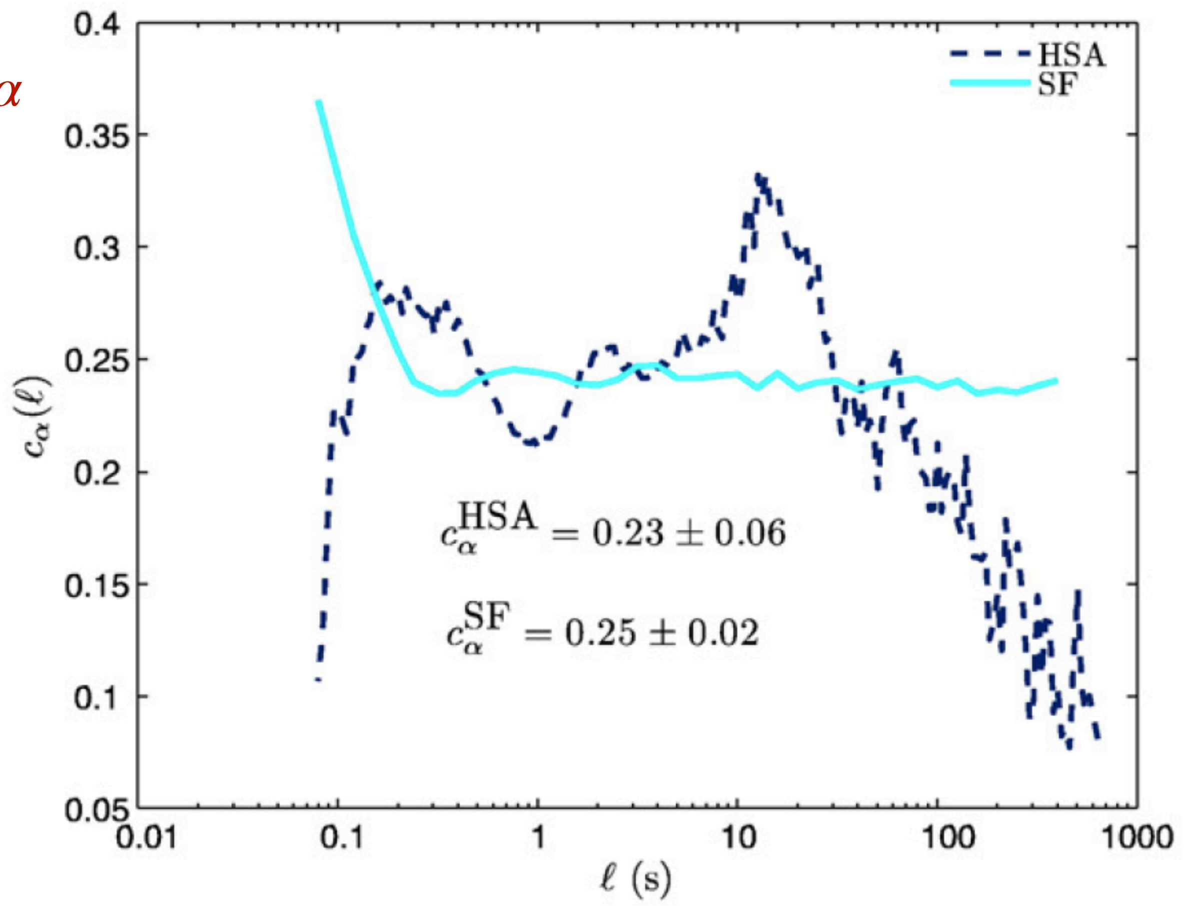
$\alpha$




Scale

Evolution versus scale of the two parameters, for surf-zone oceanic velocity data

$C_\alpha$




Scale

# Large moments?

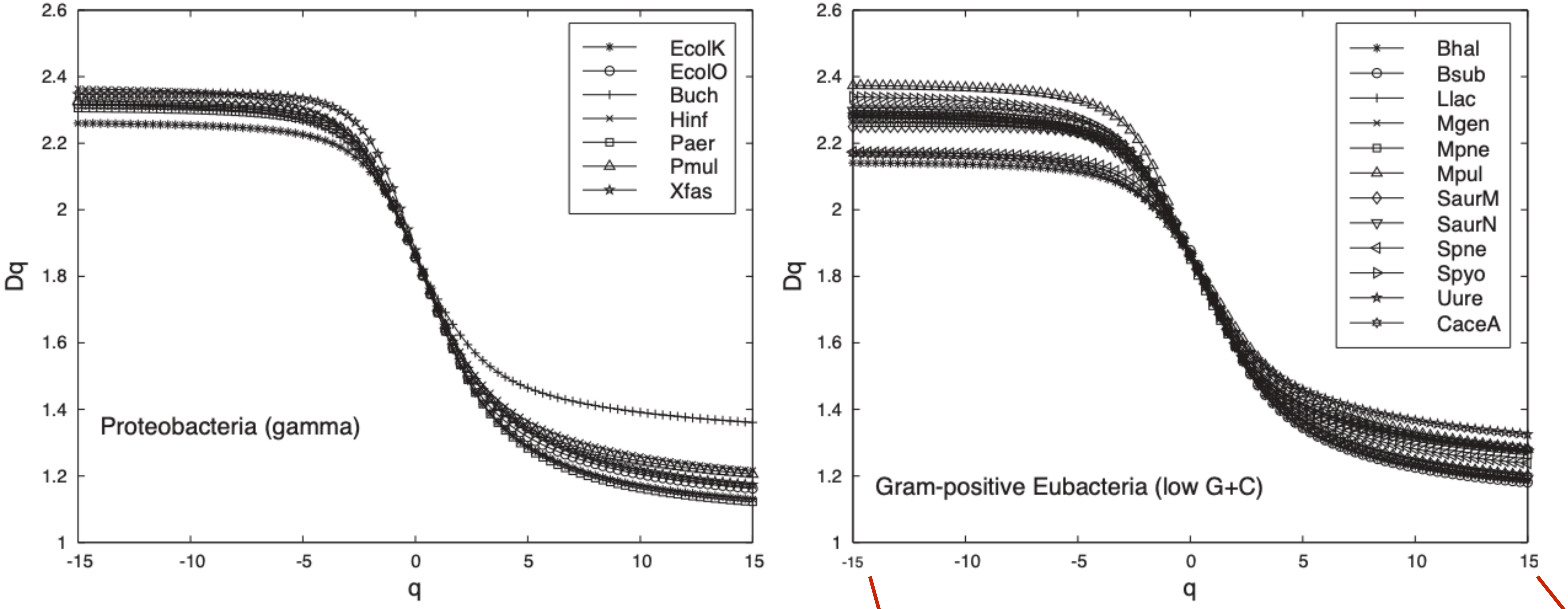



Fig. 3. Dimension spectra of measure $\mu$ from the CGRs of protein sequences of some organisms.

Extraction of figures from a paper (chosen randomly) using multifractal analysis of protein sequences

much too large order of positive moment

negative moments emphasize very small values… precision limitation of the data?

# Divergence of moments of cascades

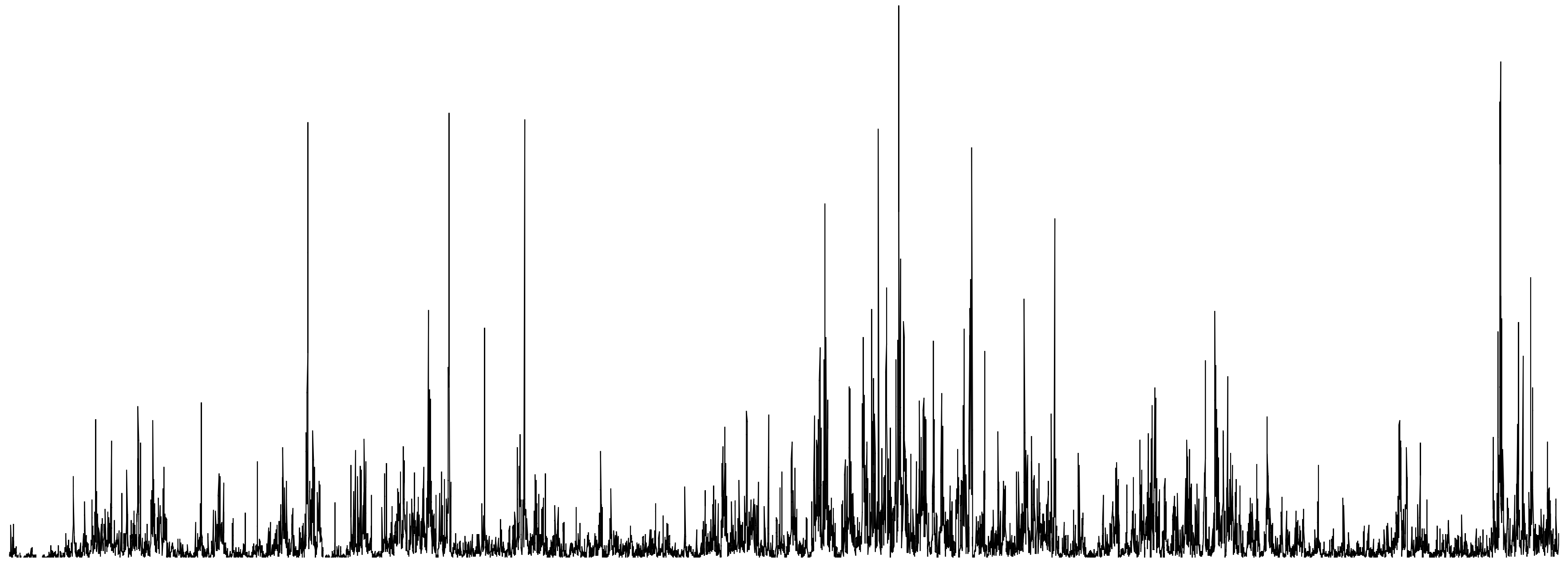
In the maths literature: « Gaussian multiplicative chaos »

$$m(A) = \int_A \underbrace{\boxed{e^{X(x)-(1/2)E[X(x)^2]}}}_{}^{\epsilon} dx$$

Robert & Vargas, 2010

$$\epsilon_\ell(x) = \frac{1}{vol(B_\ell)} \int_{B_\ell(x)} \epsilon(x')dx'$$

$$< (\varepsilon_\ell)^q >= C\ell^{-K(q)}$$



equal for $q = q_D$

It is proven that moments are infinite if: $< (\varepsilon_\ell)^q >= \infty \Longleftrightarrow K(q) \geq (q-1)d$

This means a power-law tail of the pdf: $p_{\epsilon_\ell}(x) \sim x^{-(q_D+1)}$

and also for the survival function $F(x) = \Pr(X \geq x) \sim x^{-q_D}$

Hyperbolic law

Fat tail; heavy tail

Pareto law

Frechet law

**Physics:**

Mandelbrot, 1974

Schertzer et Lovejoy, 1987

**Maths:**

Peyrière et Kahane, 1976

Kahane, 1985

Guirvarc'h, 1987

…

Robert & Vargas, 2010

+ many recent works

*The Annals of Probability*
2010, Vol. 38, No. 2, 605–631
DOI: 10.1214/09-AOP490
© Institute of Mathematical Statistics, 2010

GAUSSIAN MULTIPLICATIVE CHAOS REVISITED

BY RAOUL ROBERT AND VINCENT VARGAS

# Divergence of moments of cascades

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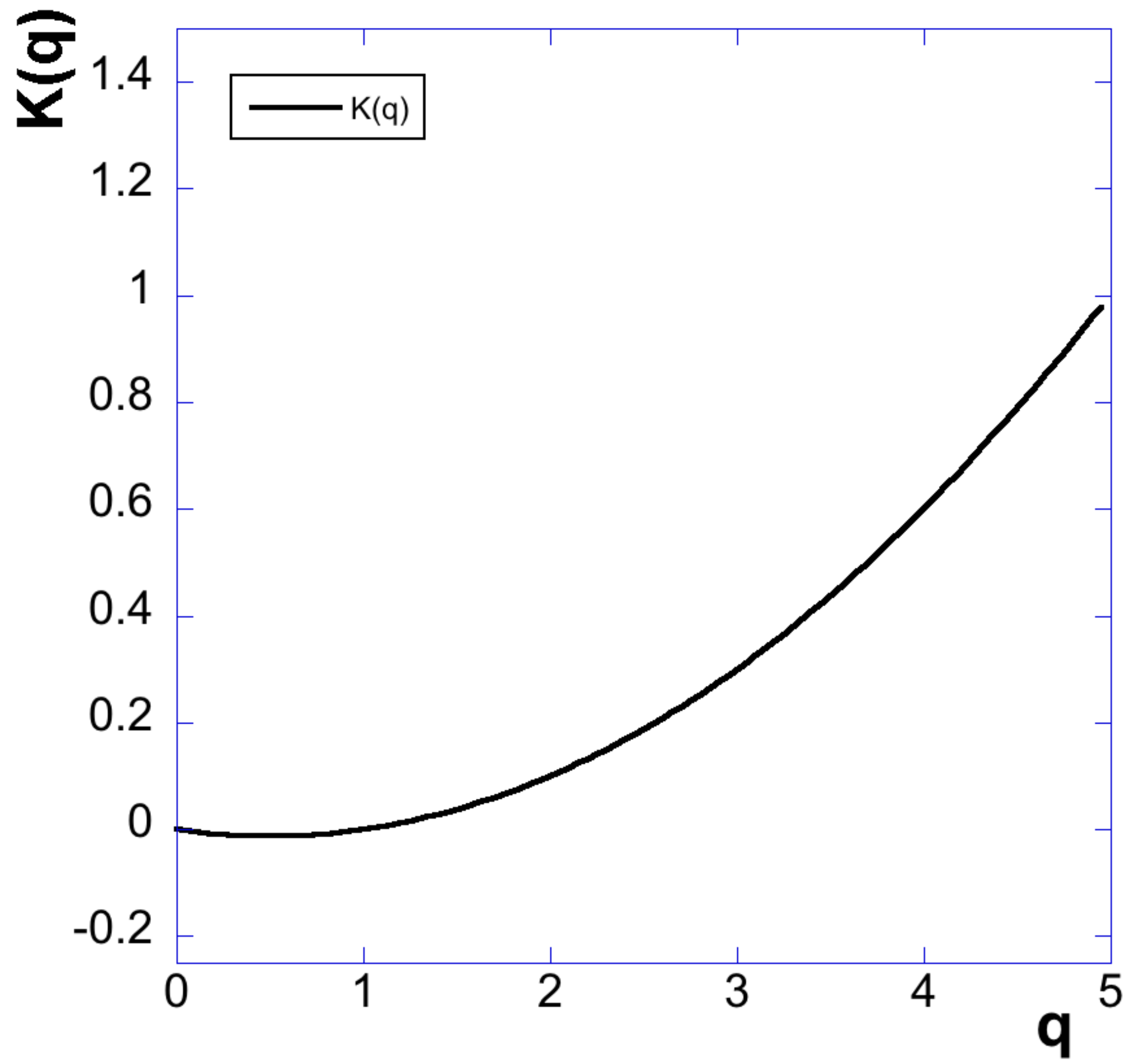
For moments larger than the threshold $q_D$, experimental estimates are not **infinite**, but their value depend on the sampling

$$K_e(q) = \begin{cases} K(q), & q \leq q_D \\ h_e q - \Delta_s, & q > q_D \end{cases}$$




Schertzer and Lovejoy, 1992

Schmitt, F., D. Schertzer, S. Lovejoy, and Y. Brunet: Empirical study of multifractal phase transitions in atmospheric turbulence, *Nonlinear Processes in Geophysics*, 1, 2/3, 95-104, 1994

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Muzy et al, 2006

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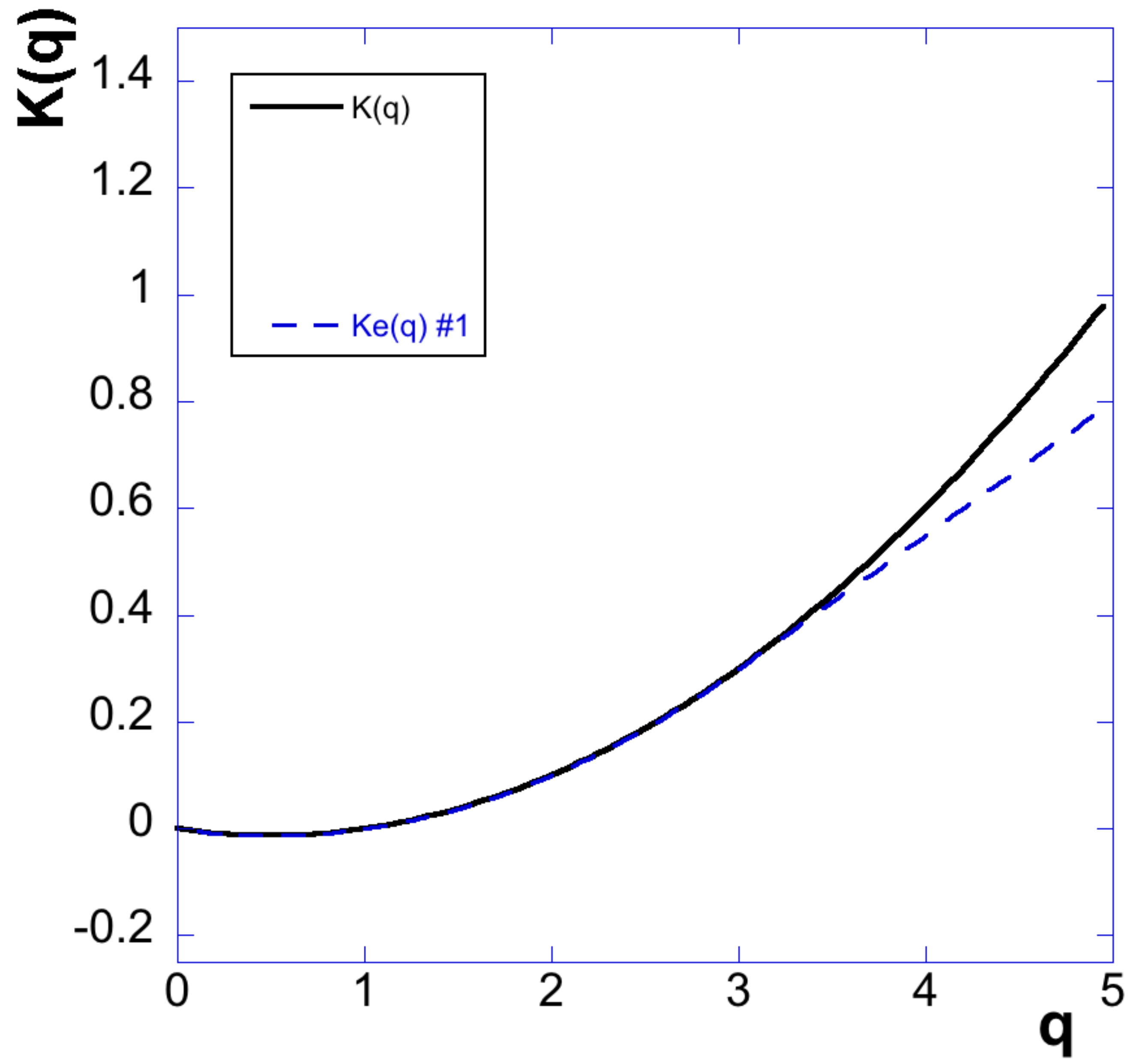



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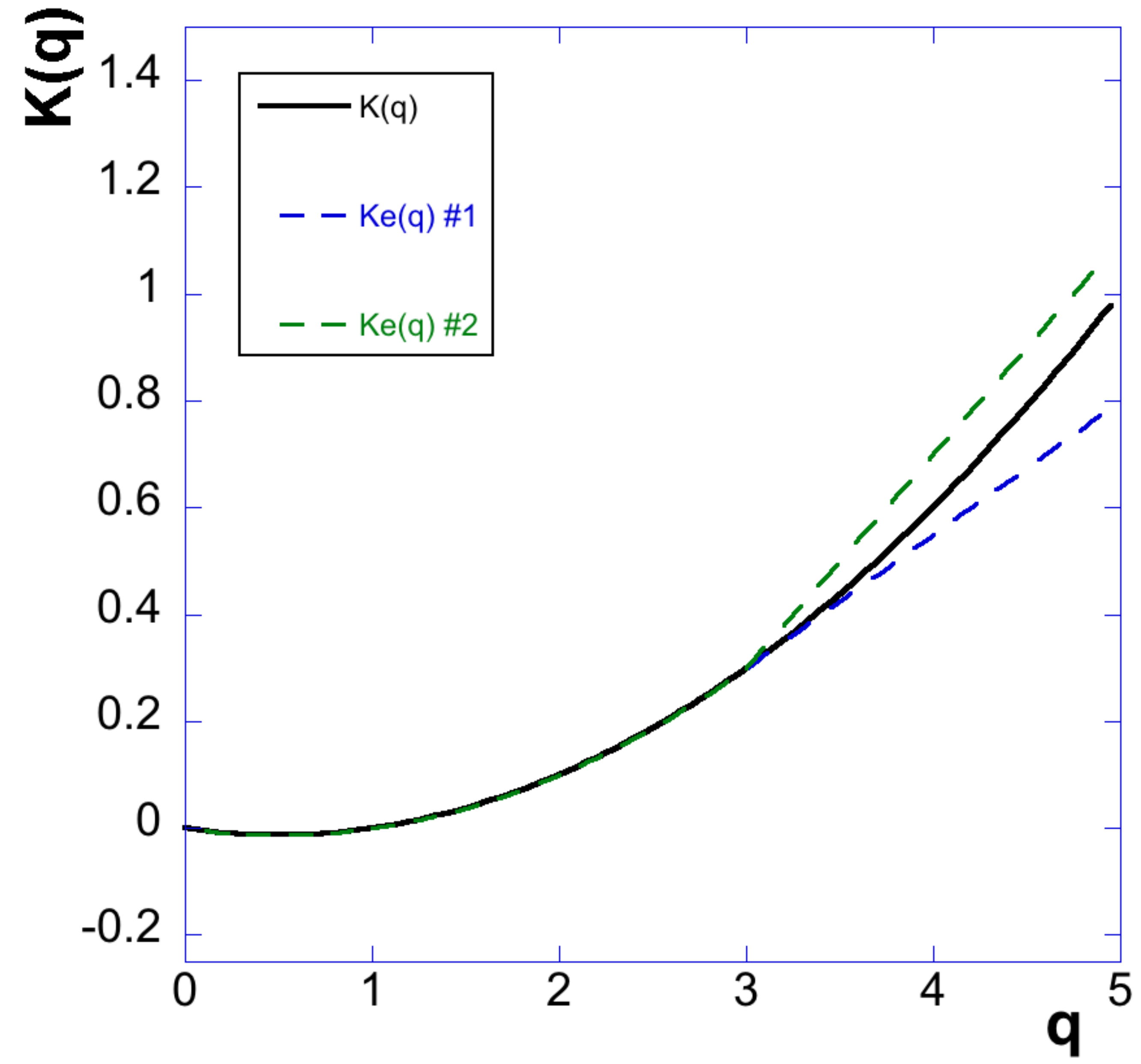
Muzy et al, 2006

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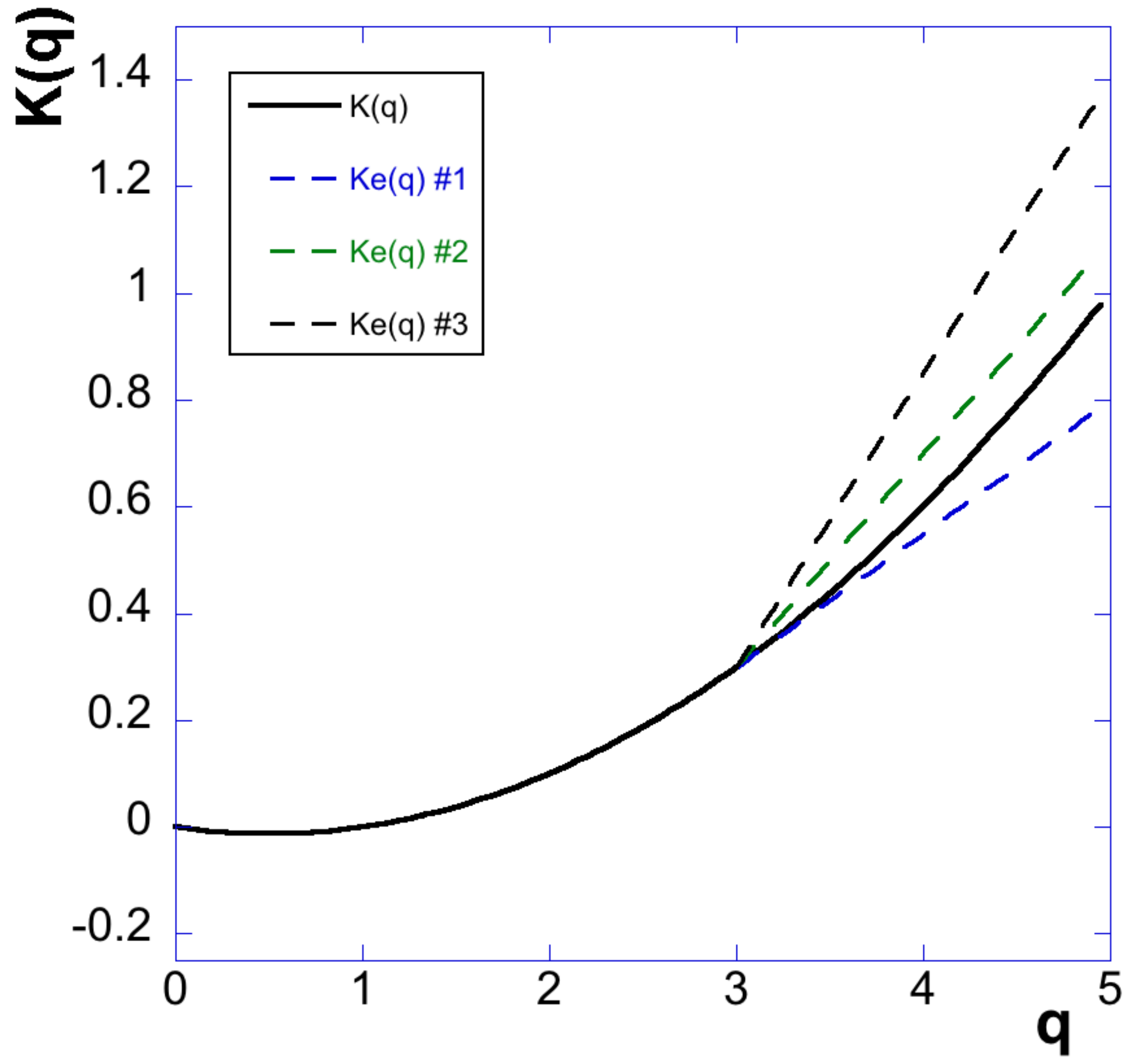



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Muzy et al, 2006

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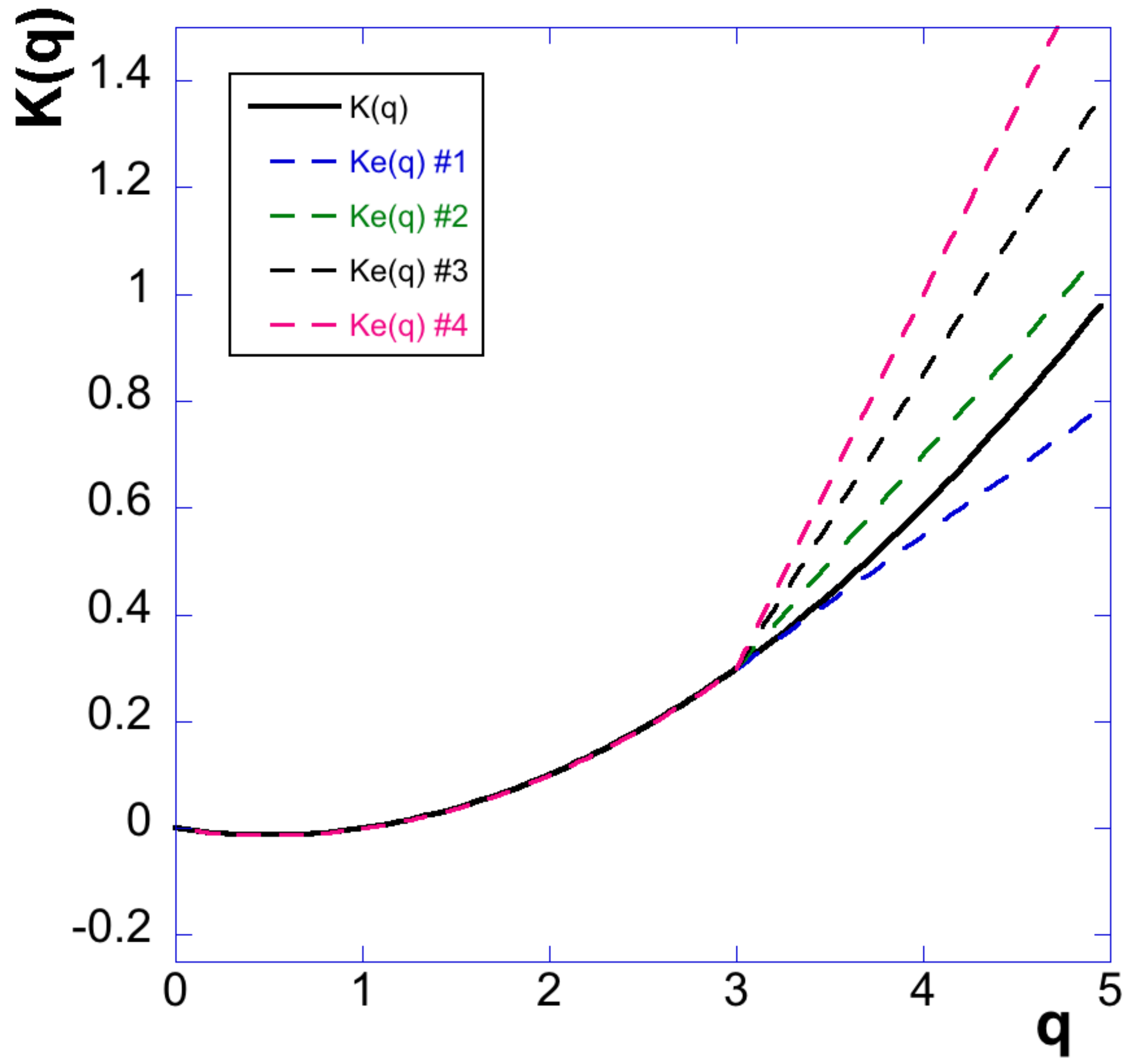
Muzy et al, 2006

# Divergence of moments for velocity structure functions?

$$\Delta V_\ell \propto (\varepsilon_\ell)^{1/3}\ell^{1/3}$$

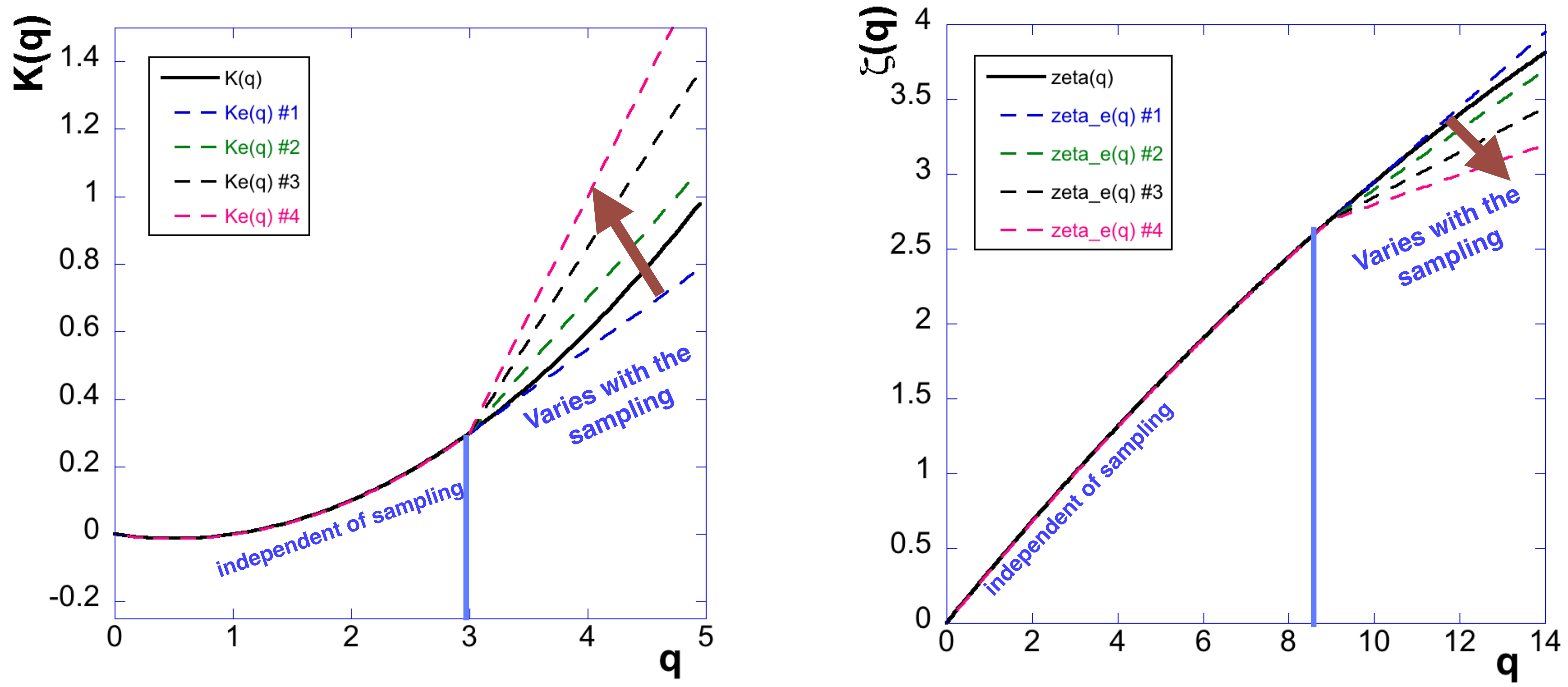
$$< (\Delta V_\ell)^{3q} > \sim < \epsilon_\ell^q > \ell^q$$

Due to this K62 relationship, a divergence of the order $\alpha$ for the dissipation corresponds to a divergence of order $3\alpha$ for velocity increments

$$< (\Delta V_\ell)^q > = \infty \quad q \geq q_v = 3q_D$$

What is theoretically expected:

$$\zeta_e(q) = \begin{cases} \zeta(q), & q \leq q_v = 3q_D \\ \Delta_s + \frac{q}{3}(1-h_e), & q > q_v \end{cases}$$




Above the critical order of moments, the estimated value changes with the sampling size

Schertzer and Lovejoy, 1992

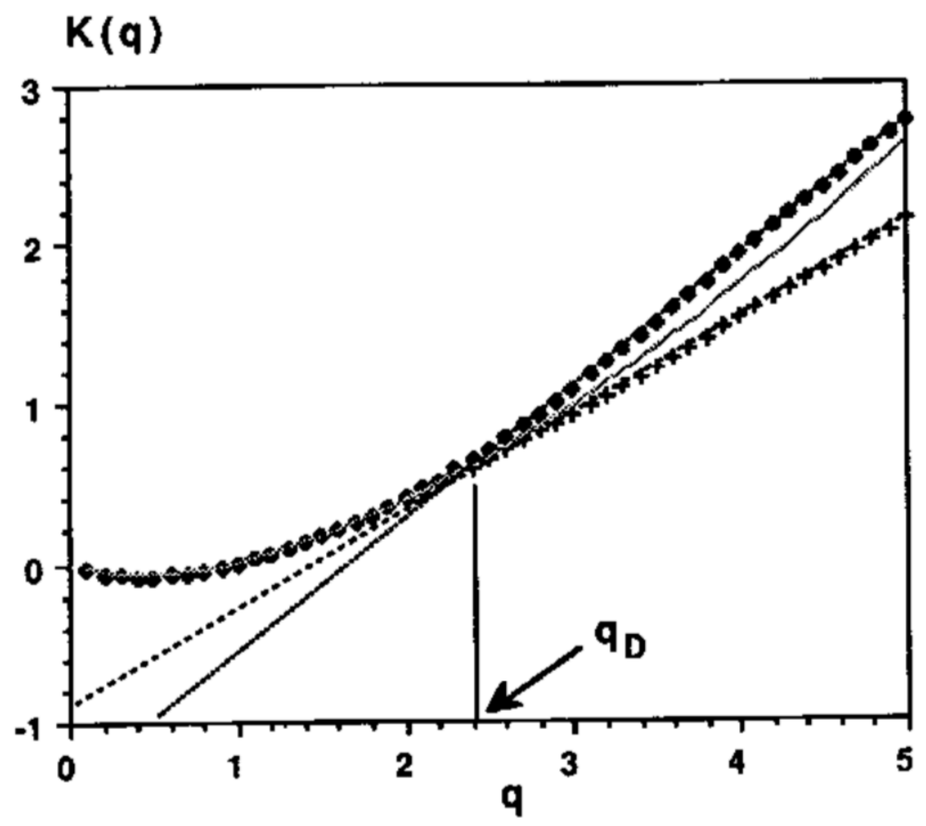
Schmitt, F., D. Schertzer, S. Lovejoy, and Y. Brunet: Empirical study of multifractal phase transitions in atmospheric turbulence, *Nonlinear Processes in Geophysics*, 1, 2/3, 95-104, 1994

Schertzer, D., S. Lovejoy, et F. Schmitt: Structures in turbulence and multifractal universality, in *Small-scale structures in 3D hydro and MHD turbulence*, ed. M. Meneguzzi, A. Pouquet and P.L. Sulem, Springer Verlag, 137-144, 1995

…

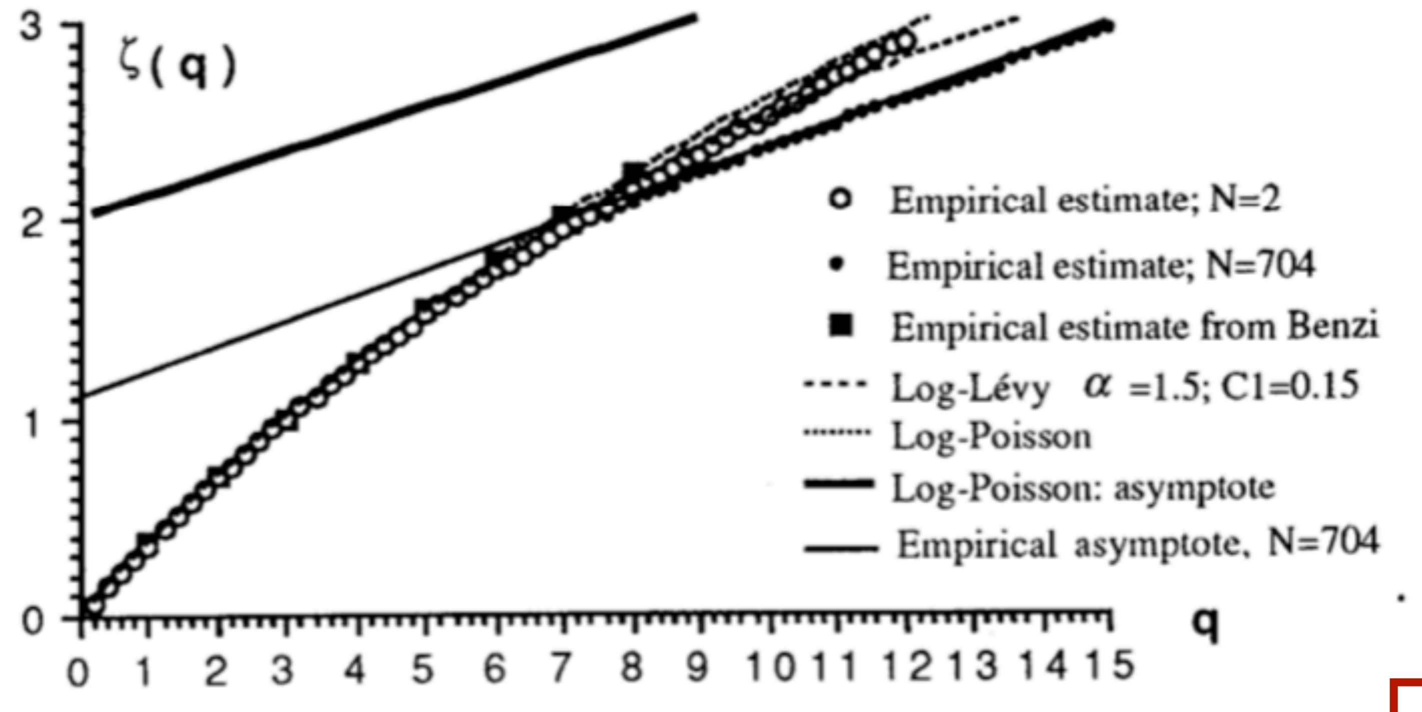
Muzy et al, 2006

# Divergence of moments for velocity structure functions?




$$q_D \simeq 2.4$$

Dissipation

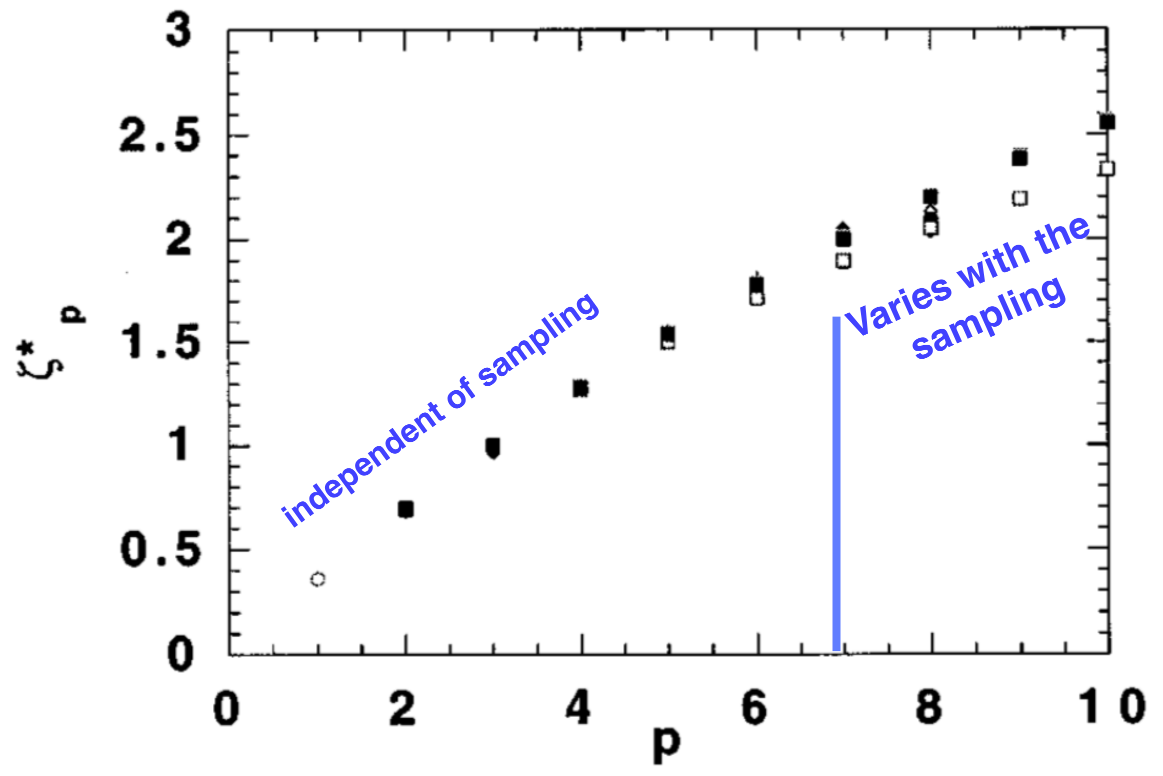



$$q_v \simeq 7$$

Velocity increments

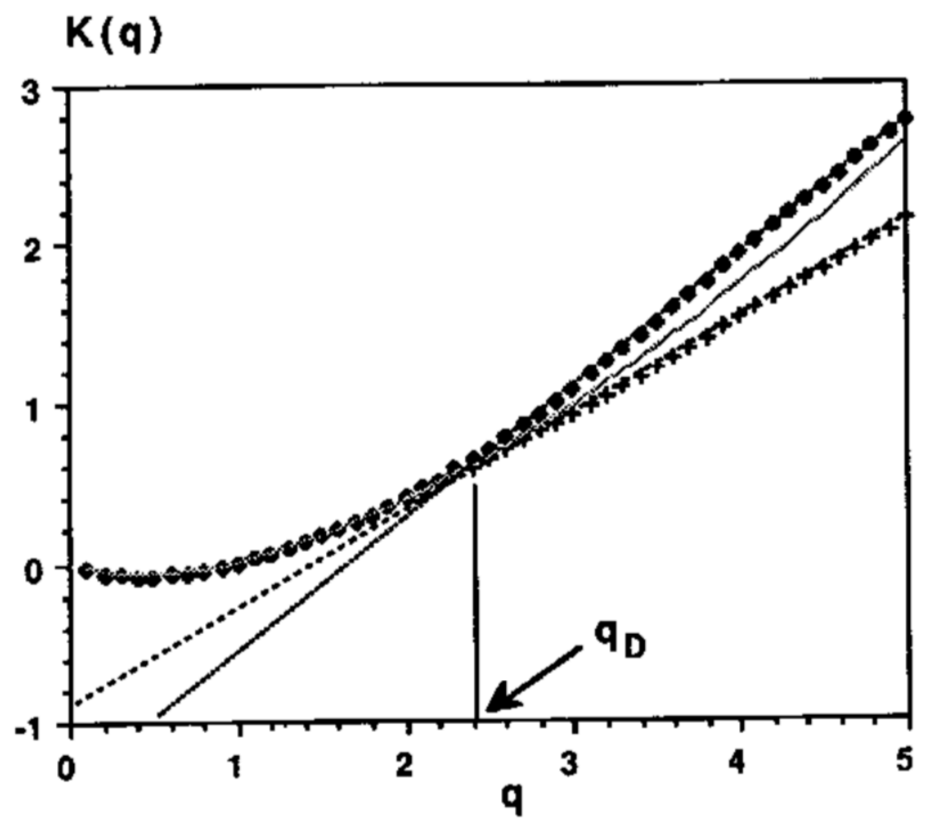
Schmitt, F., D. Schertzer, S. Lovejoy, and Y. Brunet: Empirical study of multifractal phase transitions in atmospheric turbulence, *Nonlinear Processes in Geophysics*, 1, 2/3, 95-104, 1994

Arneodo et al. Structure functions in turbulence, in various flow configurations, at Reynolds number between 30 and 5000, using extended self-similarity, *Europhysics Letters*, 34, 411-416, 1996.



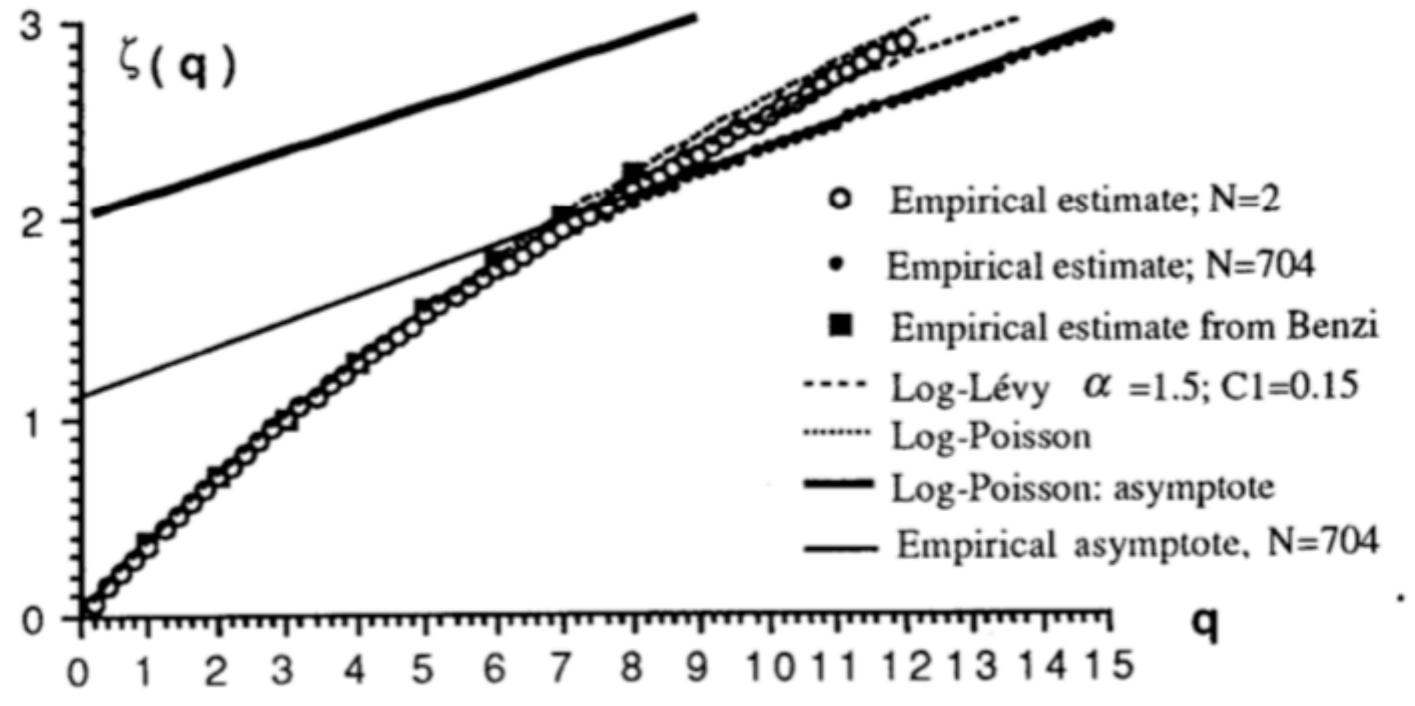

On October 7th 1994, a meeting was held between various European groups involved in experimental studies of 3D homogeneous turbulence. The aim of the meeting was to confront results obtained independently and see whether a general consensus on some properties of the velocity structure functions could be obtained. It turned out that agreement has been obtained on several characteristics of such functions, in particular on the values of scaling exponents (determined by using the technique described below), up to order 7. The participants thought that this fact was interesting to be reported. This does not mean that all the authors of the present letter agree on the significance of the result. In this letter, we essentially report facts and do not favour any particular interpretation.

# Divergence of moments for velocity structure functions?




$q_D \simeq 2.4$

Dissipation




$q_v \simeq 7$

Velocity increments

Schmitt, F., D. Schertzer, S. Lovejoy, and Y. Brunet: Empirical study of multifractal phase transitions in atmospheric turbulence, *Nonlinear Processes in Geophysics*, 1, 2/3, 95-104, 1994

Applying this criterion to our boundary layer data gives a maximum order of $q=6$, in full agreement with the previous qualitative analysis. Its application to a variety of data sets

Dudok de Wit, Can high-order moments be meaningfully estimated from experimental turbulence measurements? *Physical Review* E 70, 055302(R), 2004.

atmospheric measurements
divergence of order 7

$R_\lambda \sim 1130$

DNS data, $q_D \simeq 8$
for the velocity

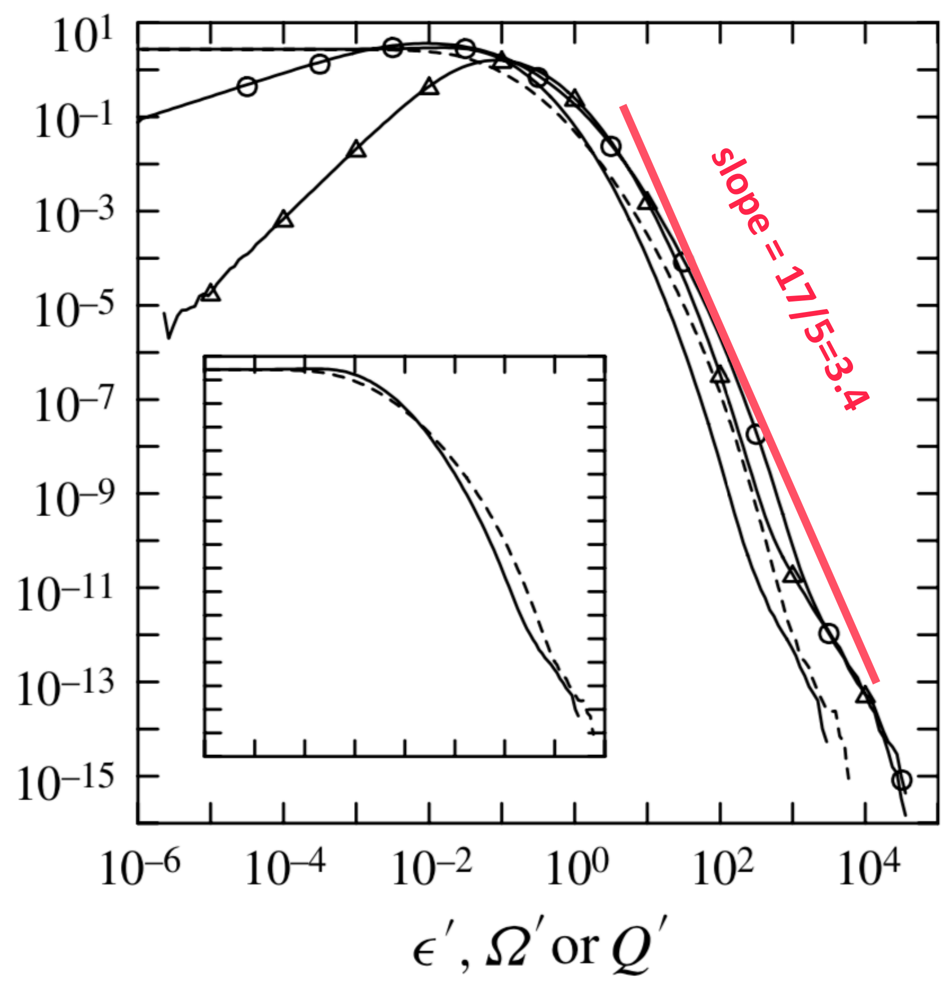
T. Ishihara, T. Gotoh, Y. Kaneda, Study of high-Reynolds number isotropic turbulence by direct numerical simulation, *Annual Review of Fluid Mechanics*, 41, 165-180, 2009.

The DNS data at high *Re* suggest that the scaling exponents $\zeta_p^L$ up to $p = 8$ are likely to be universal (insensitive to large-scale flow conditions), but it is unknown whether the universality applies at higher orders or to transverse exponents $\zeta_p^T$ or mixed exponents $\zeta_{p,q}^{LT}$.

$R_\lambda \sim 1000$

P.K. Yeung, D.A. Denzis, K. R. Sreenivasan, Dissipation, enstrophy and pressure statistics in turbulence simulations at high Reynolds numbers, *Journal of Fluid Mechanics*, 700, 5-15, 2012.

DNS data, divergence of order 2.4
for the dissipation



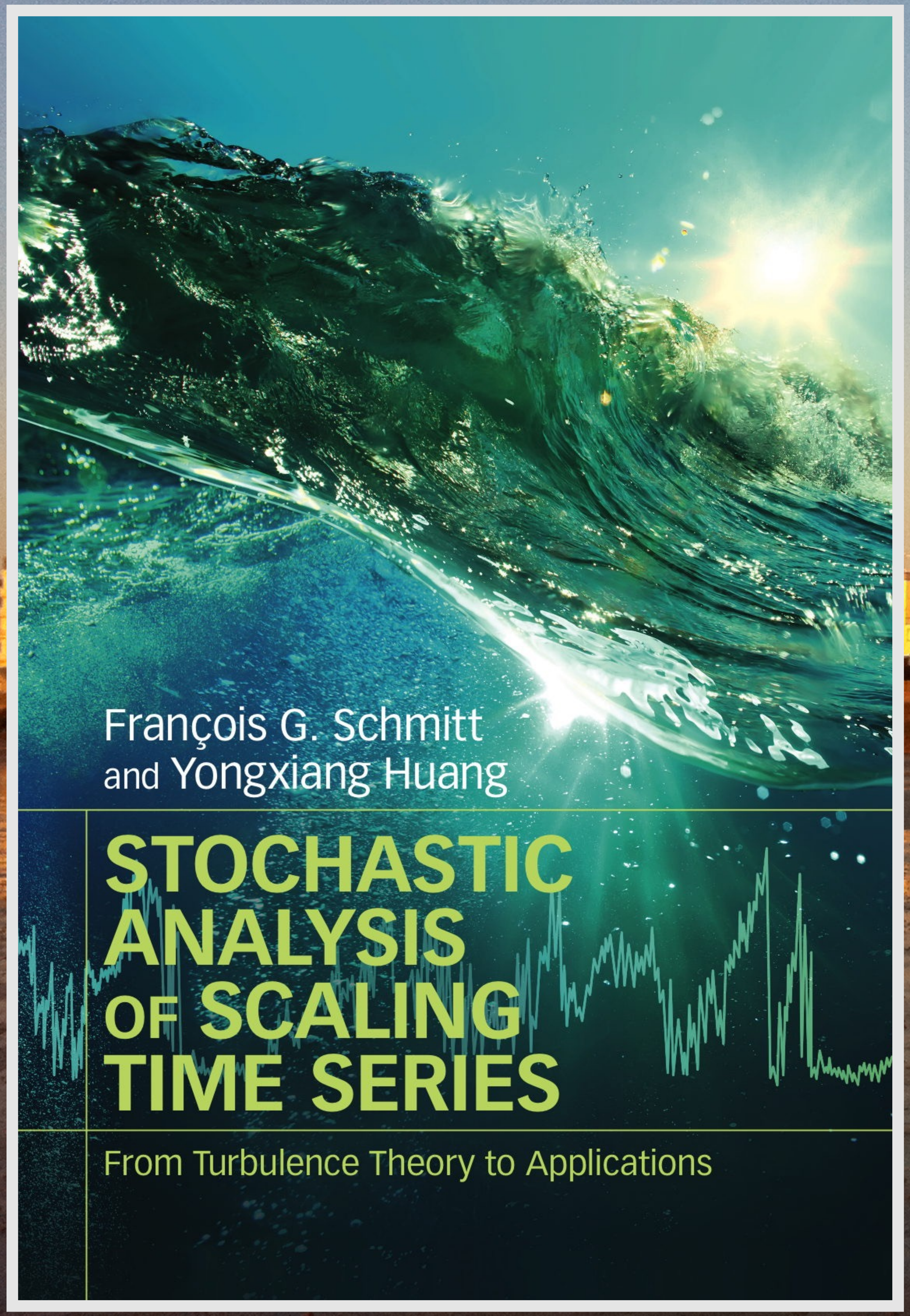
# Conclusions/Perspectives

**Conclusions:**

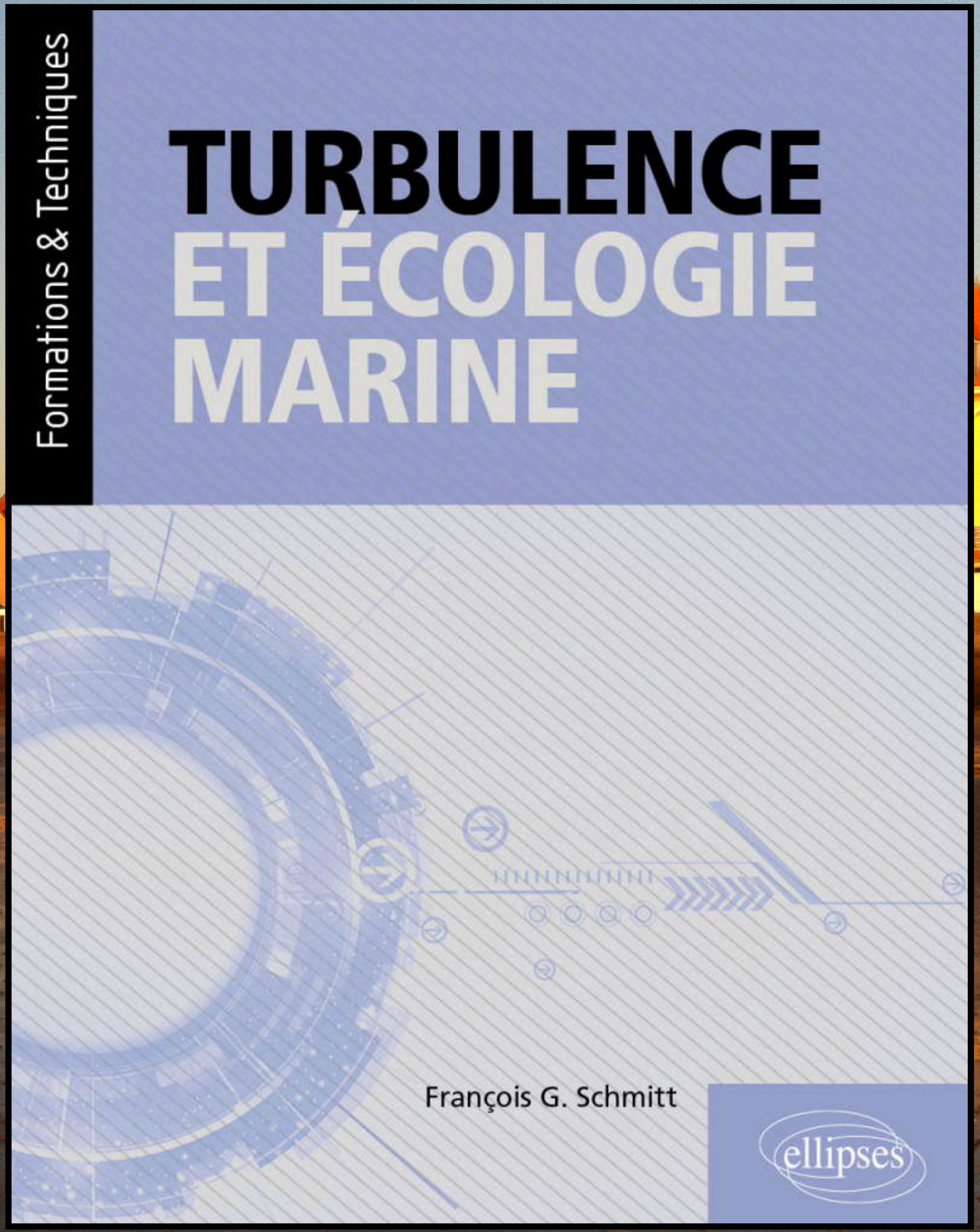
- Coarse-graining for measures.
- Structure functions, wavelets, EMD-HSA (or other methods) for non-stationary, or multi-affine fields.
- Different multifractal log-ID models exist, corresponding to different analytical expressions of the nonlinear part of the curves.
- Can be applied to many different fields.
- Limitations of the order of moment: sampling limitation, or divergence of moments. Several evidences for a divergence of moments of around 2.4 for the dissipation, and of $q_D = 7 \pm 1$ for velocity increments in turbulence

**Perspectives/what is still to be done?**

- Turbulence: what relations between Navier-Stokes equations (deterministic) and the multifractal and intermittent properties of velocity and passive scalars?
- Cascades and intermittency: correspond to long-range properties. Closure models (eddy-viscosity, LES…) do not take this into account. Find a closure compatible with multifractals? [Schmitt, F. G.: About Boussinesq's turbulent viscosity hypothesis: historical remarks and a direct evaluation of its validity, C. R. Mécanique, 335, 617-627, 2007]
- Predictions taking into account the long-range properties of scaling and multifractal fields.
- We know how to generate continuous multiplicative cascades. But how to generate a multiaffine stochastic process (in a « clean » and general way?
- Extensions to the multi-dimensional case.




Cambridge University Press, 2016




Ellipses, 2020