École des Ponts
ParisTech

A century of cascades and multifractal operators

EGU Campfire 11/01/2022

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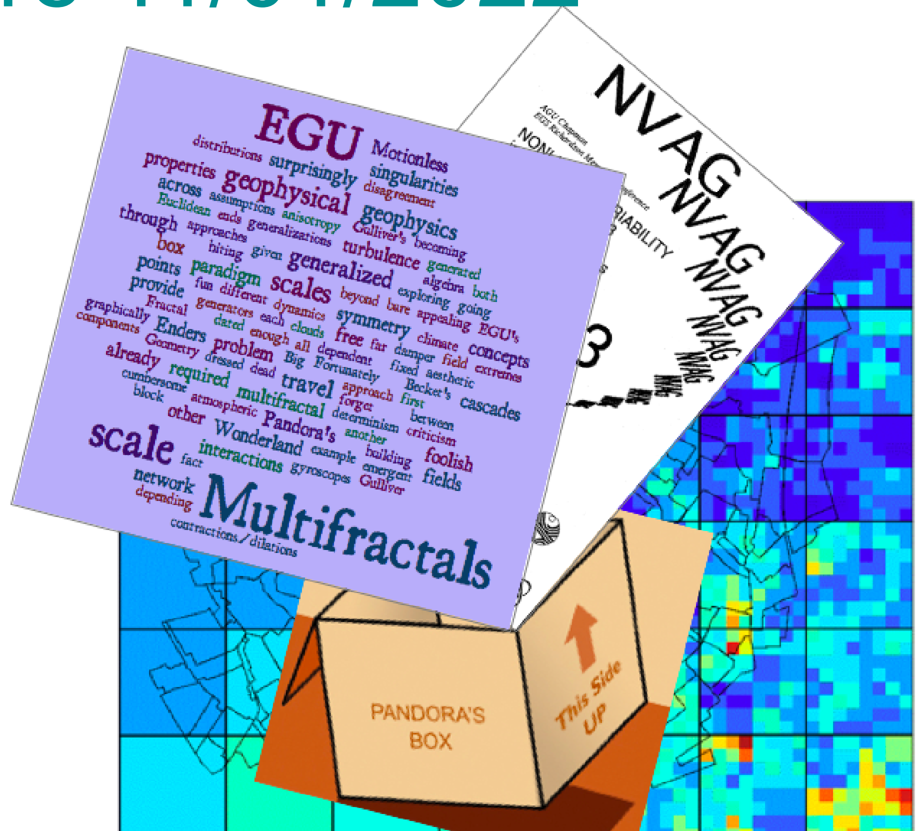

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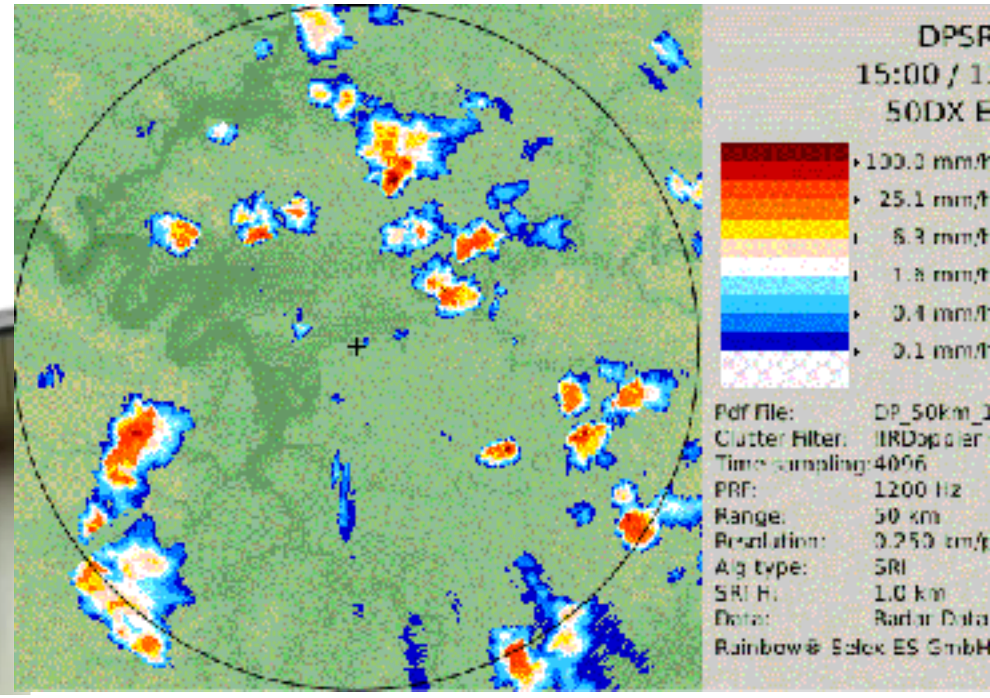
Millenium problem of turbulence !

exploratorium



Art piece 'Windswept' (Ch. Sowers, 2012): 612 freely rotating wind direction indicators to help a large public to understand the complexity of environment near the Earth surface

Polarimetric radar observations of heavy rainfalls over Paris region during 2016 spring (250 m resolution):

- **heaviest rain cells** are much smaller than **moderate ones**
- **complex dynamics** of their aggregation into a large front

Two centuries … since Navier (1822)

Louis Navier
(1822)

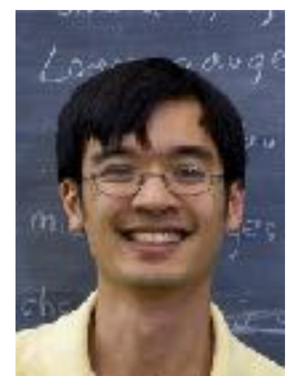
Augustin-Louis
Cauchy

Adhémar Jean Claude
Barré de Saint-Venant

Sir George Stokes
(1843)

A **millenium problem** raised at Ecole Nationale des Ponts et Chaussées, recent episodes:

Otelbaev (2013) and Tao (2015)



A century of cascades !

WEATHER PREDICTION

BY

NUMERICAL PROCESS

BY

LEWIS F. RICHARDSON, B.A., F.R.MET.SOC., F.INST.P.

FORMERLY SUPERINTENDENT OF ESKDALEMUIR OBSERVATORY
LECTURER ON PHYSICS AT WESTMINSTER TRAINING COLLEGE

We realize thus that: big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity—in the molecular sense.

The Supply of Energy from and to Atmospheric Eddies.

By LEWIS F. RICHARDSON.

(Communicated by Sir Napier Shaw, F.R.S. Received March 9, 1920.)

Introduction.

This paper extends Osborne Reynold's theory of the Criterion of Turbulence, to make it apply to the case in which work is done by the eddies, acting as thermodynamic engines in a gravitating atmosphere. For

From scaling analysis to cascade processes

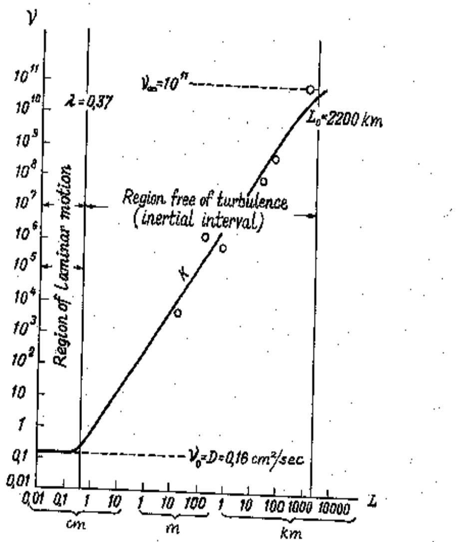
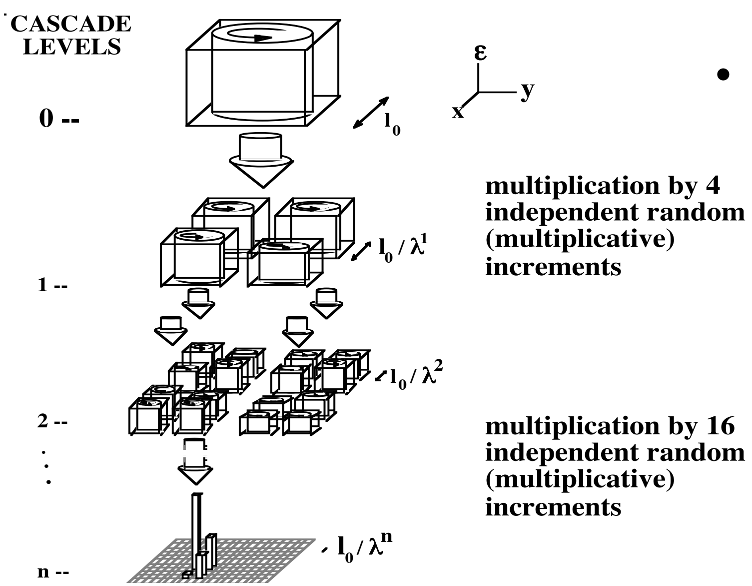


Fig. 1 The vertical diffusion coefficient $\nu(L)$ as a function of the turbulence scale L. Empirical points after Richardson.[19]

Richardson, 1926



Schertzer & Lovejoy, 1989b

- Scaling analysis
  - Passive scalar dispersion: Richardson (1926)
  - Structure function: Kolmogorov (1941)
  - Energy spectrum: Obukhov (1941)
  - Higher order structure functions: Kolmogorov (1962), Obukhov (1962)
  - Renyi dimensions: Grassberger and Procaccia (1983), Hentschel,and Procaccia (1983)
  - Legendre transform to dimensions: Parisi and Frisch (1985)
  - Fractal measures: Halsey et al. (1987)
  - ….
- Cascade processes
  - ß-model: Novikov and Stewart (1964), Frisch et al. (1978)
  - Log normal model: Yaglom (1966)
  - Limit log-normal: Mandelbrot (1974)
  - a-model: S+L (1984, 1985)
  - Multiplicative chaos: Kahane (1985)
  - Universal multifractals/Levy multiplicative chaos: S+L (1987a&,b, 1997)), Fan (1987)
  - Log-Poison model: Dubrulle (1974)
  - ….

Varenna summer school (1983)

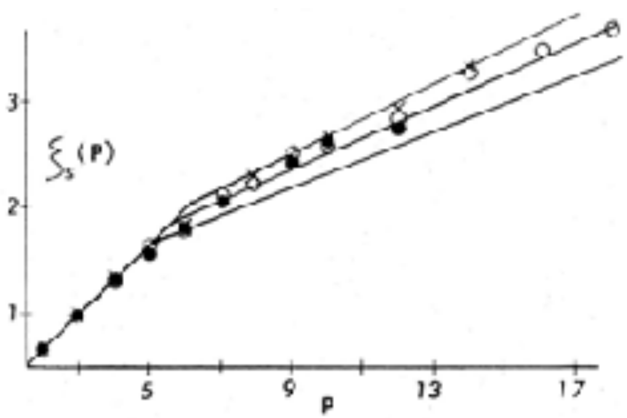
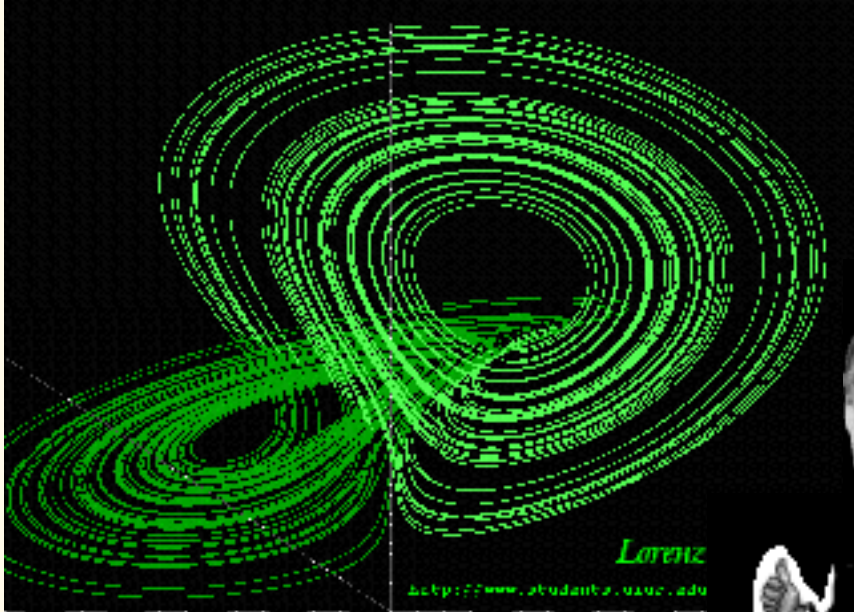


Fig.1 from S+L (1984)

- "Turbulence and Predictability in Geophysical Fluid Dynamics" organised by M. Ghil, R. Benzi et G. Parisi
  - a primary version of the multifractal formalism of Parisi and Frisch (1985) was presented there after many informal discussions. This paper concludes by: "**Still the multifractal model appears to be somewhat more restrictive than Mandelbrot's weighted-curdling model which does include the logornormal case**".
  - we had the opportunity to deliver a short formal presentation (the conference proceedings (1985) refers to S+L (1984)):
    - a small perturbation of the ß-model is no longer limited to a unique dimension ($\alpha -$model)
    - the divergence of higher order moments is rather generic in cascade models
    - the later can introduce spurious scaling, an analytical approximation depending on a unique scaling exponent H and the critical order α was proposed:

$$\xi(p) = pH + \theta(p-\alpha)(1 - p/\alpha)$$

    - it was shown to fit the experimental points from Anselmet et al. (1983), see fig. 1 with $H = 1/3, \alpha = 5, 5.5, 6$



Ed's statement to cool down a hot debate (Varenna, 1983)

2008 AGU
Lorenz lecture

Cascades and statistical physics

Transformation of a measure σ with the help of a "density" ε into another measure Π :

$$d\Pi = \varepsilon d\sigma$$

Generalisation with a non trivial limit ε of densities ε_λ of increasing resolution: $\lambda = L/\ell \to \infty$

$$\varepsilon_\lambda = e^{\Gamma_\lambda};\ \mathrm{E}e^{q\Gamma_\lambda} = \mathrm{Z}_\lambda(q) = e^{\mathrm{K}_\lambda(q)} \approx \lambda^{K(q)} \overset{\text{Mellin}}{\longleftrightarrow} \mathrm{P}(\varepsilon_\lambda > \lambda^\gamma) \approx \lambda^{-c(\gamma)}$$

$$\downarrow$$

Legendre

$$K(q) \longleftrightarrow c(\gamma)$$

main "trick":
log-divergence of the generator

Γ: generator ≈ hamiltonian

q : statistical order ≈inverse of temperature

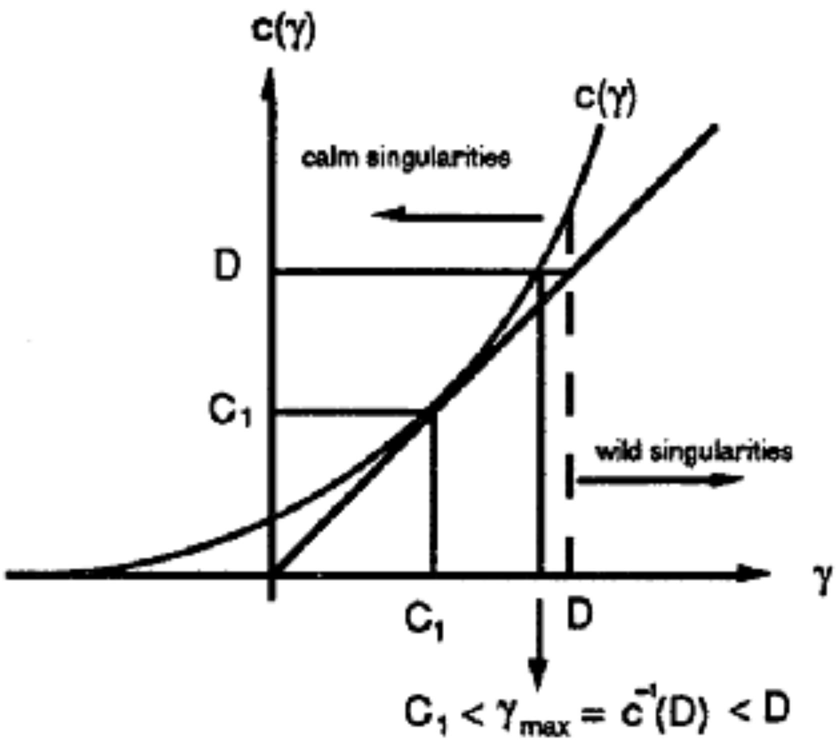
Z: 1st characteristic or moment generating function ≈ partition function

K: 2nd characteristic or cumulant generating function ≈ Gibbs free entropy

c: codimension or Kramer function ≈ entropy

γ: singularity or Hölder exponent ≈ energy

Codimension vs. Dimension formalisms



$$dim(A) + codim(A) = D$$

- codimensions easier for stochastic processes (S+L 85, 87, & 88, M88, 92 (Kramer functions))

- convergence vs. degenerescence independent of the domain dimension

upper dim(σ) $< C_1 \Rightarrow \varepsilon\sigma = 0$ (degenerescence)
lower dim(σ) $> C_1 \Rightarrow \mathrm{E}\varepsilon\sigma = \sigma$ (conservative)
$\Rightarrow \mathrm{E}\varepsilon$ is a projector

- relations between deterministic dimensions and stochastic codimensions:

$$\alpha_D + \gamma = D = f(\alpha_D) + c(\gamma)$$

$$D(q) + C(q) = D; \tau_D(q) = (q-1)D(q); K(q) = (q-1)C(q)$$

Universality

- Strong statistical universality: stable Lévy variables

$$\forall n \in N, \exists a(n), b(n) \in R : \boxed{\sum_{i=1}^{n} X_i =^d a(n)X + b(n)}$$

$$\exists \alpha \in (0,2] : a(n) = n^{1/\alpha}; \alpha < 2, \forall s \gg 1 : \mathrm{P}(|X| > s) \approx s^{-\alpha}$$ (hyperbolic/Pareto tail)

$\alpha = 2 :$ Gauss

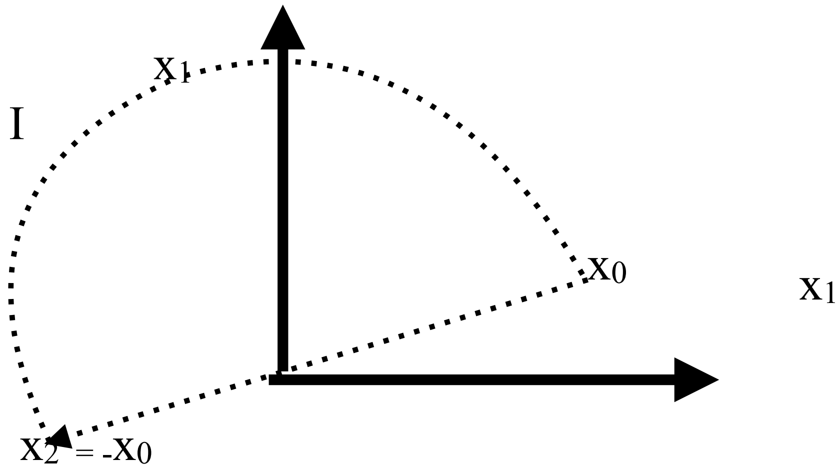
A stable Levy *X* is attractive for any Y_i having same type of tail:

$$\boxed{\lim_{n \to \infty} \frac{\sum_{i=1}^{n} Y_i - b(n)}{a(n)} =^d X}$$

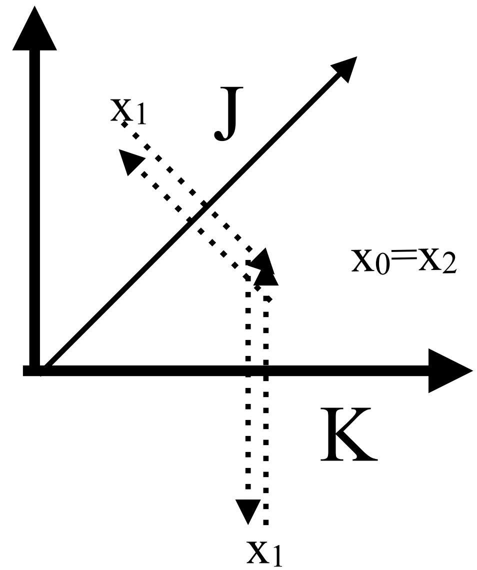
Log-Levy e^{qX}: its moments $E(e^{qX})$ are finite for any q >0, iff it has only a negative Pareto tail, i.e. iff *X* is an extremely asymmetric/skewed Lévy stable

Symmetries and unity roots

I=rot (π/2)



J, K =mirror symmetries



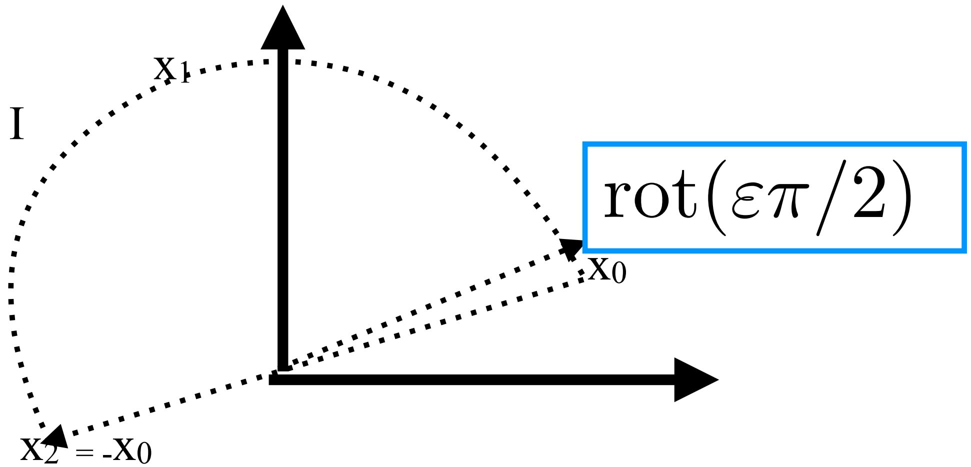
$I^2=-1$

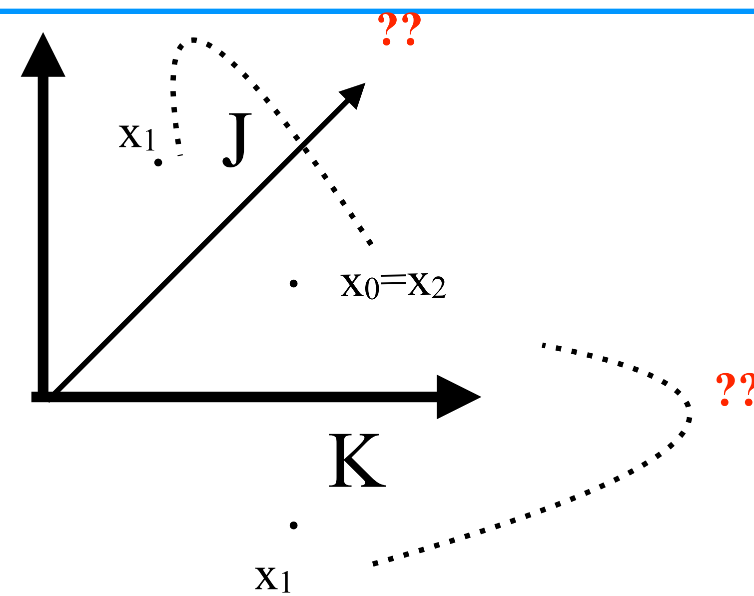
$K^2 = J^2 = 1$

Symmetries and unity roots

I=rot ($\pi/2$)

J, K =mirror symmetries





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Symmetries and unity roots

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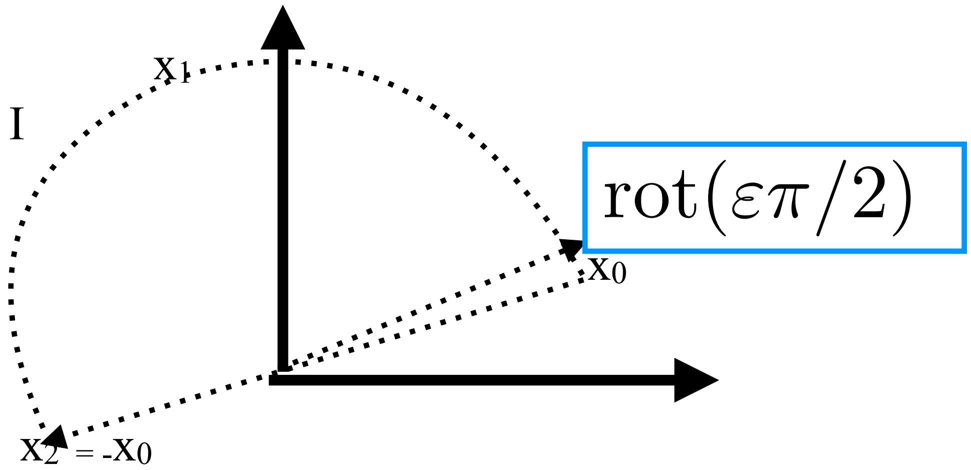

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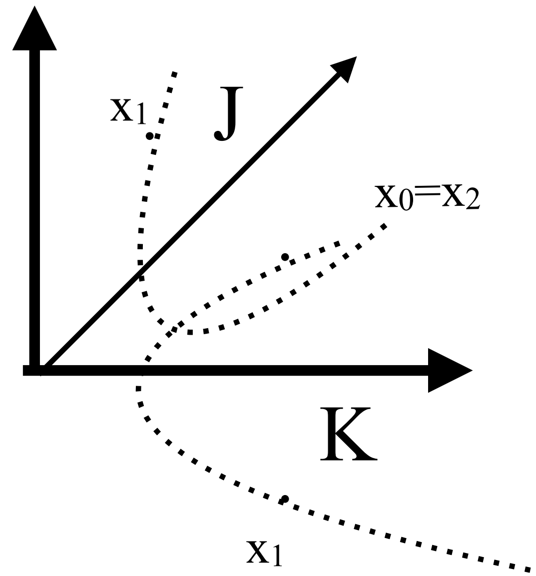

$I^2=-1$

$K^2=J^2=1$

Spherical geometry —> Hyperbolic geometry

Combining symmetries

2D linear Lie algebra H'= *l*(2, R):

$$G = d\mathbf{1} + e\mathbf{I} + f\mathbf{J} + c\mathbf{K};$$

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$2I = [J, K], \quad 2J = [I, K], \quad 2K = [J, I]$$

anti-commutators:

$$\{I, J\} = \{J, K\} = \{K, I\} = 0$$

$$I^2 = -J^2 = -K^2 = IJK = -1$$

(pseudo or split quaternions)

"quaternion equation" (Hamilton, 16/10/1843)

$$I_2^2 = J_2^2 = K_2^2 = I_2 J_2 K_2 = -1$$

Combining symmetries

2D linear Lie algebra H'= $l(2, R)$:

$$G = d\mathbf{1} + e\mathbf{I} + f\mathbf{J} + c\mathbf{K};$$

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

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$$2I = [J, K], \quad 2J = [I, K], \quad 2K = [J, I]$$

anti-commutators:

$$\{I, J\} = \{J, K\} = \{K, I\} = 0$$

$$I^2 = -J^2 = -K^2 = IJK = -1$$

$$I_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; J_2 = \begin{bmatrix} 0 & -K \\ K & 0 \end{bmatrix}; K_2 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

“quaternion equation” (Hamilton, 16/10/1843)

$$I_2^2 = J_2^2 = K_2^2 = I_2 J_2 K_2 = -1$$

Algebra of cascade generators

- Clifford algebra, dimension = 2^n
  - real numbers *R* (n=0), complex numbers *C* (n=1), quaternions *H* (n=2) other hyper-complex numbers, external algebras and many more!
- $Cl_{p,q}$: generated by operators $\{e^i\}$ that anti-commute and square to plus or minus the identity:

$$e^i e^j = -e^j e^i \ (i \neq j) \qquad (e^i)^2 = \pm 1$$

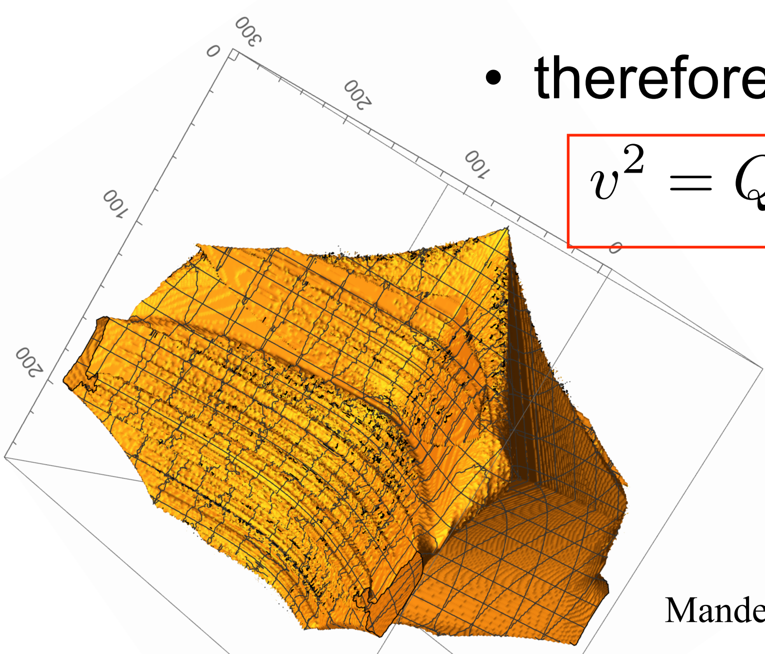
- therefore a quadratic form *Q* of signature (p,q, $p+q=n$):

$$v^2 = Q(v)1 \quad Q(v) = v_1^2 + v_2^2 .. + v_p^2 - v_{p+1}^2 - v_{p+2}^2 .. - v_{p+q}^2$$

ex.: $R = Cl_{0,0}$; $C = Cl_{0,1}$; $H = Cl_{0,2}$

H'= $I(2, R) = Cl_{2,0} = Cl_{1,1}$

"pseudo-/split- quaternions"



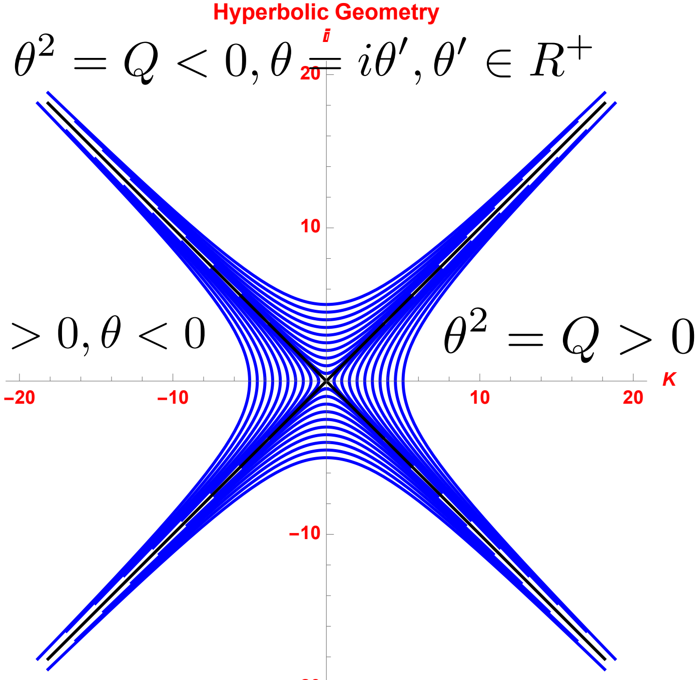
Mandelbrot set on pseudo-quaternions, S&T,2018

From algebra to group

{K, I}

$\theta^2 = Q < 0, \theta = i\theta', \theta' \in R^+$

$\theta^2 = Q > 0, \theta < 0$

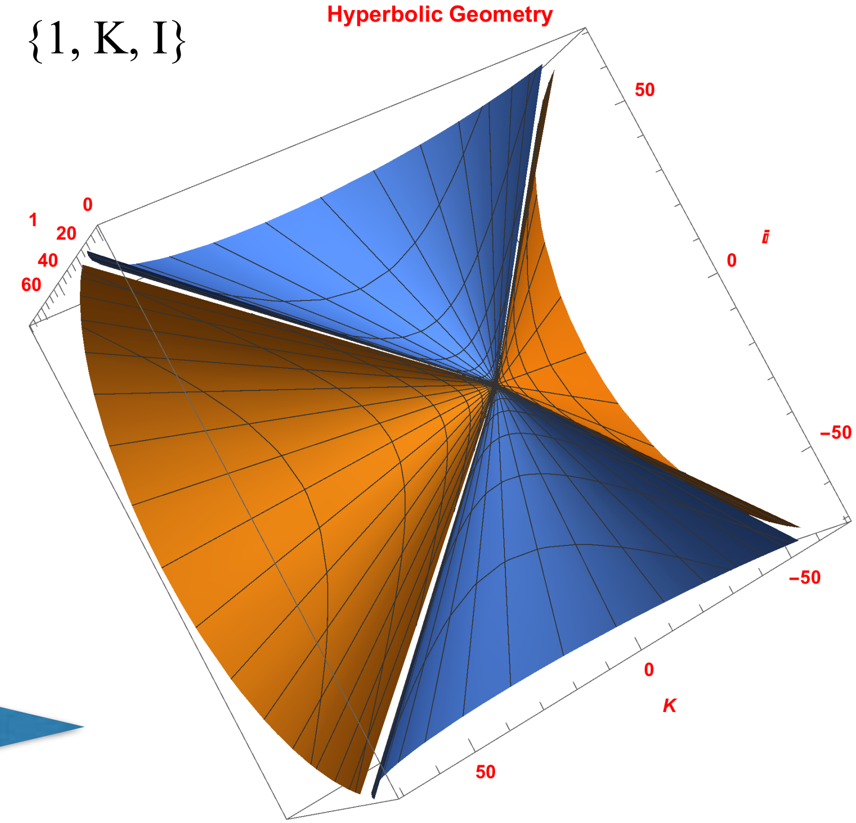


$\theta^2 = Q > 0, \theta > 0$

$\theta^2 = Q < 0, \theta = i\theta', \theta' \in R^-$

Exp

{1, K, I}



Generalised Moivre-Euler formula: $(e^{u\theta})^{\alpha} = \cosh(\alpha\theta)1 + \sinh(\alpha\theta)u$

infinite number of u, $u^2 = \pm 1$!

Stochastic Clifford?

- Statistical universality: stable Lévy vectors

$$\forall n \in N, \exists a(n), b(n) \in R : \boxed{\sum_{i=1}^{n} X_i =^d a(n)X + b(n)}$$

$$\exists \alpha \in (0,2] : a(n) = n^{1/\alpha}; \alpha < 2, \forall s \gg 1 : \mathrm{P}([X| > s) \approx s^{-\alpha}$$ (hyperbolic/Pareto tail)

$\alpha = 2 :$ Gauss

A stable Levy *X* is attractive for any Y_i having same type of tail:

$$\boxed{\lim_{n\to\infty} \frac{\sum_{i=1}^{n} Y_i - b(n)}{a(n)} =^d X}$$

– classical "quasi- scalar" case: only *b is a* vector like X_i and Y_i

– 'real' vector case: *a* and α are matrices *(S. et al., 2001)*

Exponentiation of Lévy-Clifford algebra

- Existence ?
  - Q defines a bilinear form < . >

$$< X, Y > = \frac{1}{2}\left(Q(X+Y) - Q(X) - Q(Y)\right)$$

  - which defines a Laplace-Clifford transform,
  - hence a second characteristic function (cumulant generating function

$$\mathrm{E}exp(< q, \Gamma_\lambda > = Z_\lambda(q) = exp(K_\lambda(q))$$

finite over $\mathscr{A}^{\downarrow}$

the opposite cone to that supporting the extremely assymetric Lévy stable component $\mathscr{A}^{\uparrow}$

Fractionnaly Integrated Flux model (FIF, vector version)

FIF assumes that both the renomalized propagator G_R and force f_R are known:

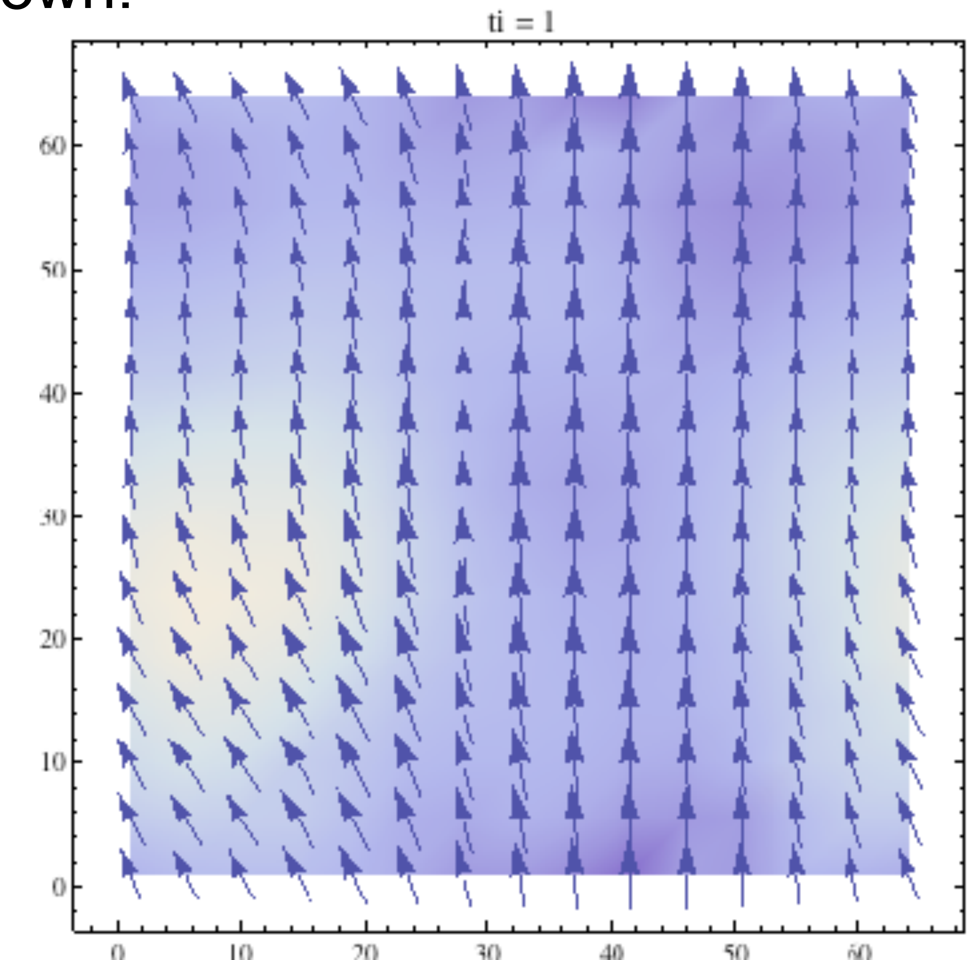
$$G_R^{-1} * u = f_R$$

where: $f_R = \varepsilon^a$

G_R^{-1} is a fractionnal differential operator

ε results from a continuous, vector, multiplicative cascade (Lie cascade)

Complex FIF simulation of a 2D cut of wind and its vorticity (color)

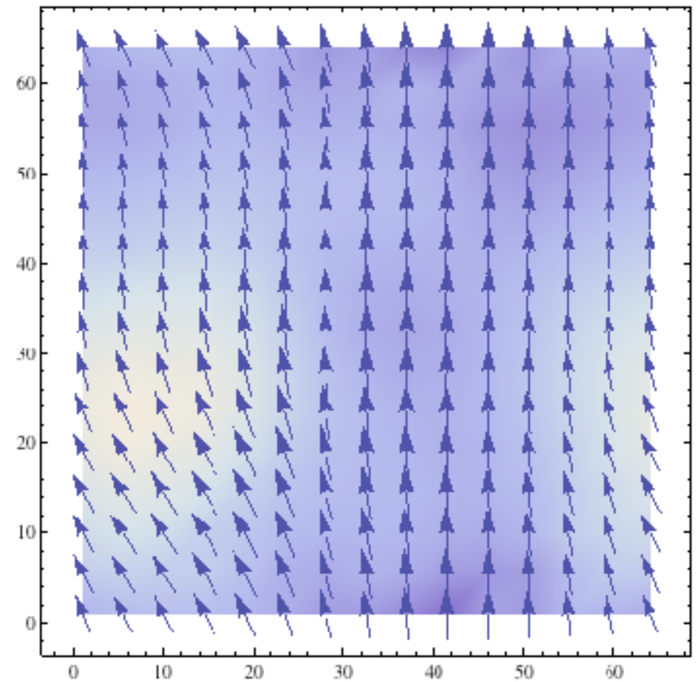


Surface layer complexity!

exploratorium®

Art piece ‘Windswept’ (Ch. Sowers, 2012): 612 freely rotating wind direction indicators to help a large public to understand the complexity of environment near the Earth surface

WAUDIT Wind resource assessment audit and standardization



Multifractal FIF simulation (S et al., 2013) of a 2D+1 cut of wind and its vorticity (color). This stochastic model has only a few parameters that are physically meaningful.

Both movies illustrate the challenge of the near surface wind that plays a key role in the heterogeneity of the precipitations... and wind energy!

Fractionnaly Integrated Flux model (FIF, vector version)

FIF assumes that both the renomalized propagator G_R and force f_R are known:

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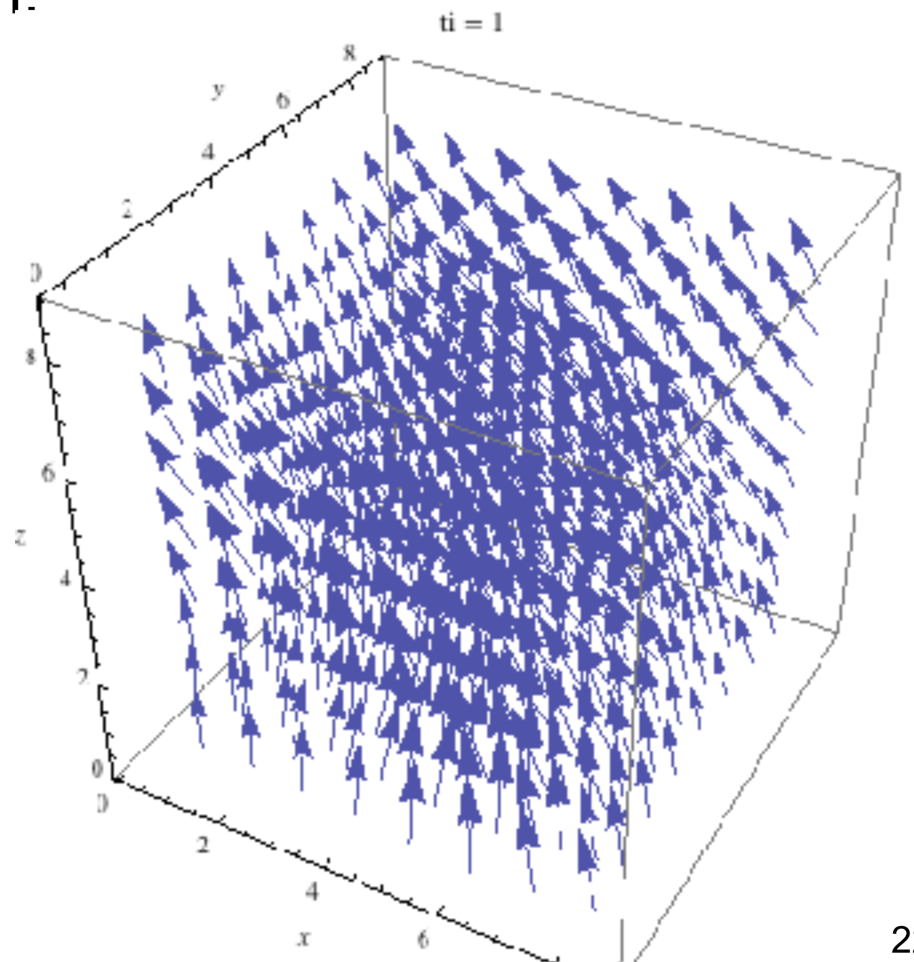
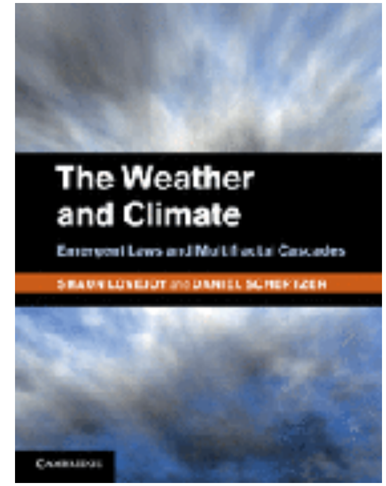
3D FIF wind simulation based on quaternions



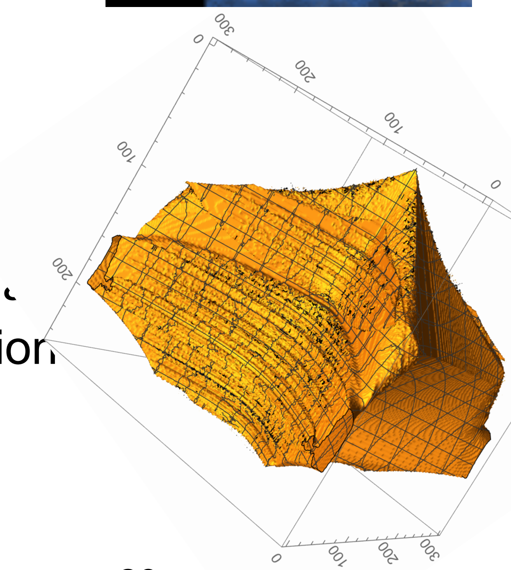
Conclusions

S&T, Earth& Space, 2020
Chaos 2015, S&al. ACP, 2012,
S&L, IJBC, 2011,
Fitton&al., JMI 2013



- Intermittency: a key issue in geophysics and a major breakthrough with multifractals in the 1980's:
  - infinite hierarchy of fractal supports of the field singularities
  - beyond commonalities significant differences of approaches and applications
- No longer limited to scalar valued fields
  - multifractal operators: exponentiation from a stochastic Lie algebra of generators onto its Lie group of transformations
  - ex. Clifford algebra $Cl_{p,q}$
  - physically meaningful and convenient to understand, analyse a simulate intermittent vector fields, more generally multidimension systems.

=> from field physics to singularity physics



Conclusions

S&T, Earth& Space, 2020
Chaos 2015, S&al. ACP, 2012,
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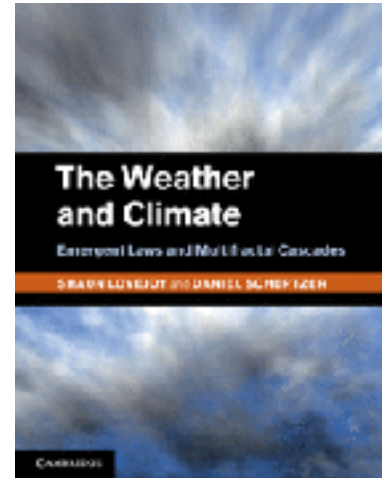
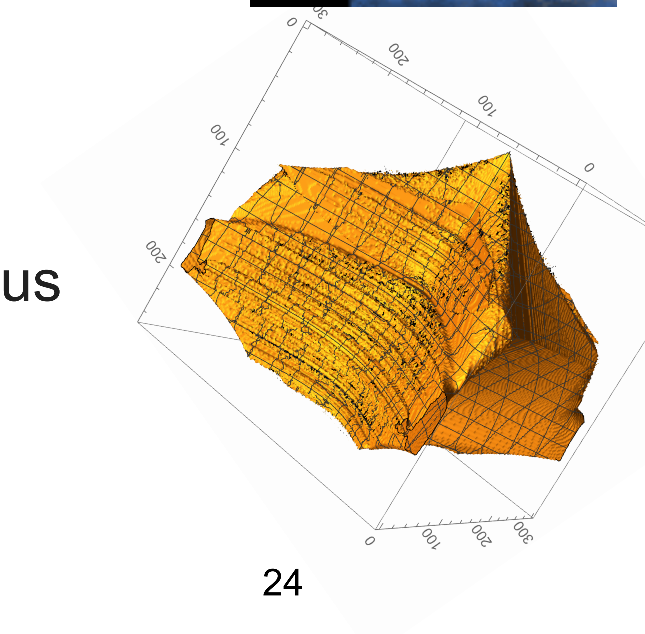
Final conclusion: the Nobel Committee for Physics is right to quote the saying reported by Philip Anderson (Phys. Today 41 526 1988):



"A real scientific mystery is worth pursuing to the ends of the Earth for its own sake, independently of any obvious practical importance or intellectual glamour."



Intermittency is without doubt such a mystery, but not without multifaceted practical importance and numerous corresponding contributions.

This is more than being illustrated by thousands of communications in EGS/EGU NP since 1988.

From geometry to analytics

T^*_λ =pullback of T_λ for functions

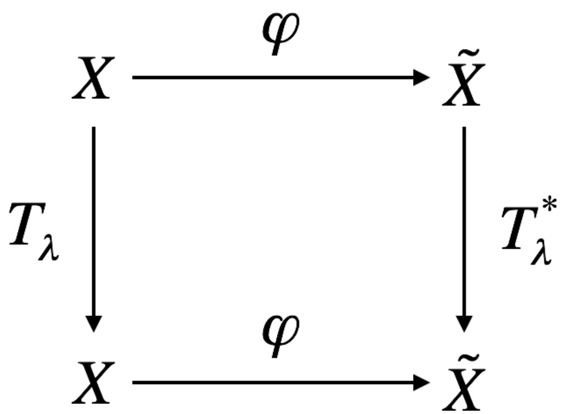


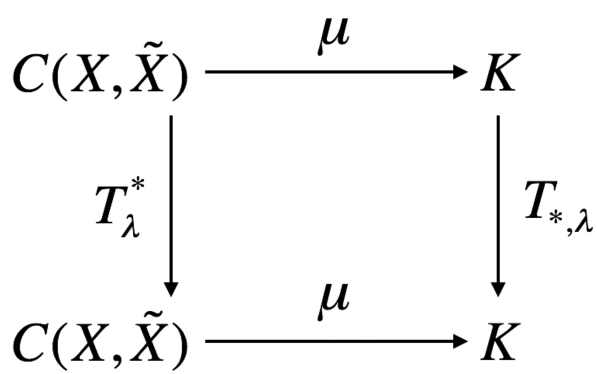
Figure 1: Commutative diagram illustrating how the analytical pullback transform T^*_λ is generated on the codomain $\tilde{X}$ of the field φ by the geometric transform T_λ on the domain X.

ex.: simple scaling (e.g. Lamperti, 1962)

$$T_\lambda x = x/\lambda,\ T^*_\lambda y = y/\lambda^H$$

$T_{*,\lambda}$ =push forward of T_λ

for measures or generalised functions



ex.: fractal measure of dimension D

$$T_\lambda x = x/\lambda,\ T_{*,\lambda}\mu = \mu/\lambda^D$$

Figure 3: Commutative diagram, similar to that of Fig. 1, illustrating how the analytical pullback transform T^*_λ generates in turn the push forward $T_{*,\lambda}$ for measures or generalized functions μ's.

From geometry to analytics

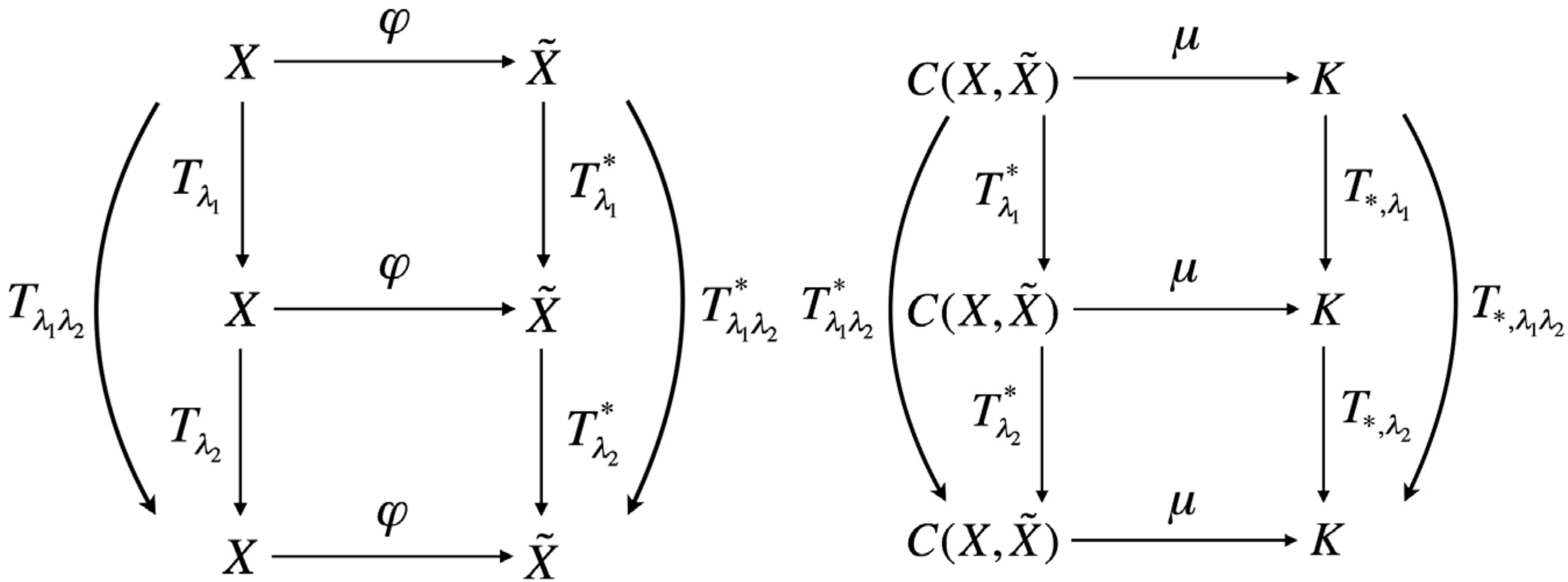
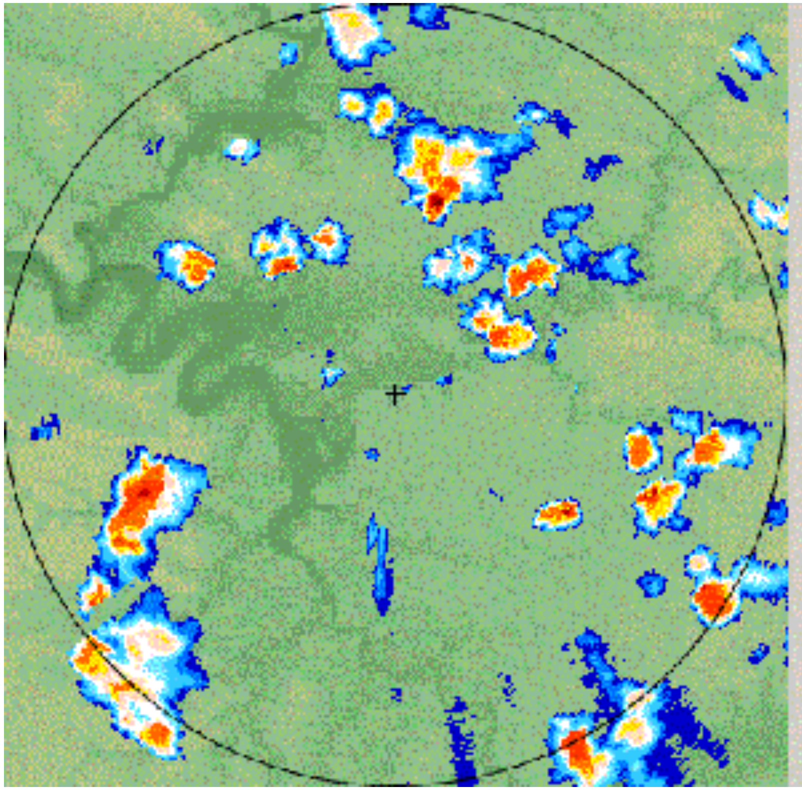


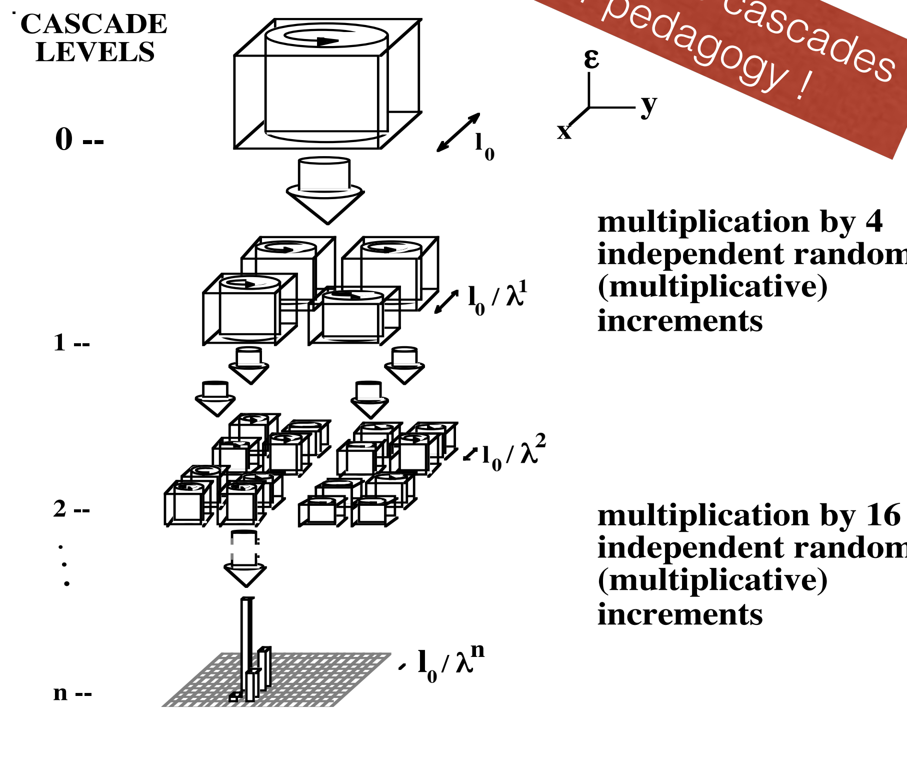
Figure 5: These diagrams show how the group property of T_λ propagates in a straightforward manner to the "pullback" transform T_λ^* (left) and then (by duality) to the "push forward" transform $T_{*,\lambda}$ (right).

Russian dolls… and multiplicative cascades

École des Ponts

Discrete in scale cascades only for pedagogy !





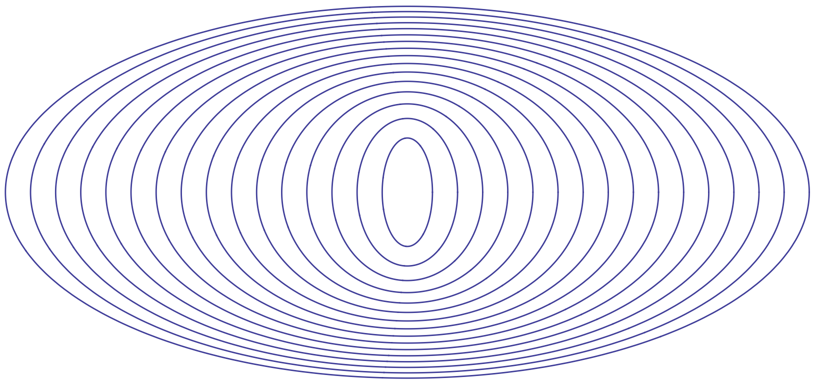
Polarimetric radar observations of heavy rainfalls over Paris region during 2016 spring (250 m resolution):

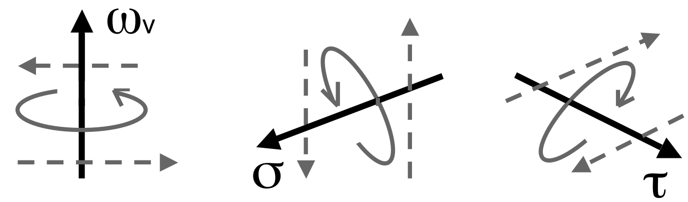
- **heaviest rain cells** are much smaller than **moderate ones**
- true for their dimensions => **multifractal field**
- **complex dynamics** of their aggregation into a large front

2+H_z-dimensional vorticity equation

(0<H_z<1)

Stratified atmosphere:





$$D\vec{\sigma}/Dt = (\vec{\sigma}\cdot\vec{\nabla}_h)\vec{u}_h$$

$$D\vec{\tau}/Dt = (\vec{\tau}\cdot\vec{\nabla}_h + \boxed{\vec{\omega}_v\cdot\vec{\nabla}_v)\vec{u}_h}$$

$$D\vec{\omega}_v/Dt = (\vec{\tau}\cdot\vec{\nabla}_h + \vec{\omega}_v\cdot\vec{\nabla}_v)\vec{u}_v$$

Strong interactions between *local generalized* scales,

= *strongly non local* (Euclidean) scales !

- a difficulty for direct numerical simulations ?
- easy for stochastic simulations !

3D Scaling Gyroscope Cascade

$$\left(\frac{d}{dt}+\nu k_n^{2}\right)\hat{u}_n^{i\,*}=i\{k_{n+1}[\left|\hat{u}_{n+1}^{2i-1}\right|^2-\left|\hat{u}_{n+1}^{2i}\right|^2]+(-1)^i k_n\hat{u}_n^{i\,*}\hat{u}_{n-1}^{a(i)}\}$$

$a(i)$ is an ancestor.

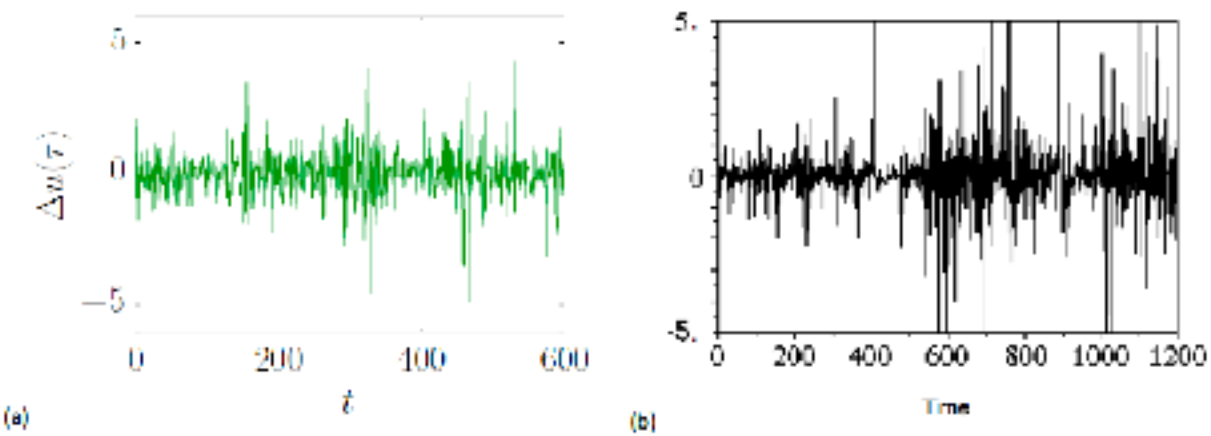
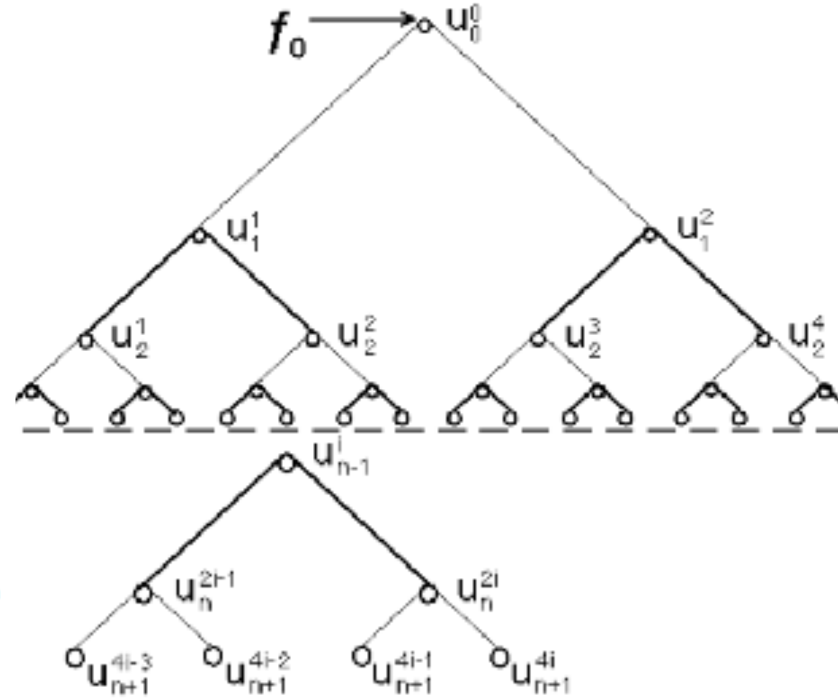


Figure 2: Comparison of fluctuations: (a) atmospheric turbulence at 100*m* (*Fitton*, 2013) and (b) SGC simulation for n 6 (*Chigirinskaya and Schertzer*, 1996), both display somehow similar strong intermittency.



Local flux of energy:

$$\varepsilon_n^i = -\sum_{r=0}^{n} k_{n-r+1}\left[\left|\hat{u}_{n-r+1}^{2a^r(i)-1}\right|^2-\left|\hat{u}_{n-r+1}^{2a^r(i)}\right|^2\right]\mathrm{Im}(\hat{u}_n^{a^r(i)})+(-1)^{a^r(i)+1}k_{n-r}\left|\hat{u}_n^{a^{r+1}(i)}\right|^2\mathrm{Im}(\hat{u}_{n-1}^{a^{r+1}(i)})$$

Mr Jourdain and Lie cascades

- Levi decomposition of any Lie algebra into its radical (*good guys*!) and a semi-simple subalgebra (*bad guys*!), e.g.:

$$l(2, R) = R\,1 \oplus_s sl(2, R)$$

What is trickier:

- large number of degrees of freedom (*dim*^{2})
- log divergence with the resolution
- universality:
  - Levy multivariates, unlike Gaussian mutivariates, are non parametric (*)
  - asymmetry of Levy noises to have convergent statistics, e.g.:

$$\forall n \in N, \forall X \geq 0 : \ \exp(X) \geq X^n/n!$$

(S&L, 95, T&S 96)

(*) limitation of anamorphosis transform and/or geostatistics

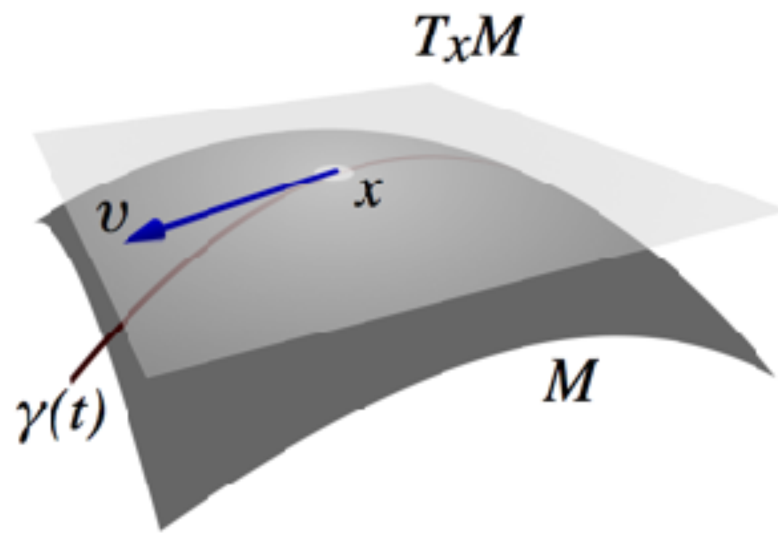
Mr Jourdain and Lie cascades

What is general and theoretically straightforward:

- $exp : Lie\ algebra \longmapsto Lie\ group$

scalar valued cascade: R^d --> R^+

- Lie group: smooth manifold
- Lie algebra: tangent space to the group at the identity
- therefore a vector space with a skew product that satisfy the Jacobi identity:



$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

exemple: commutator of matrices

$$[X,Y] = XY - YX \qquad [X,Y] = 0 \Rightarrow \exp(X+Y) = \exp(X)\exp(Y)$$

Clifford algebra

- An important family of Lie algebras of operators:
  - their dimension: 2^n
  - generalizes real numbers *R* (n=0), complex numbers *C* (n=1), quaternions *H* (n=2) and other hyper-complex numbers, external algebras and more!
- $Cl_{p,q}$ has a basis $\{e^i\}$ whose vectors anti-commute and square to plus or minus the identity:

$$e^i e^j = -e^j e^i \ (i \neq j) \qquad (e^i)^2 = \pm 1$$

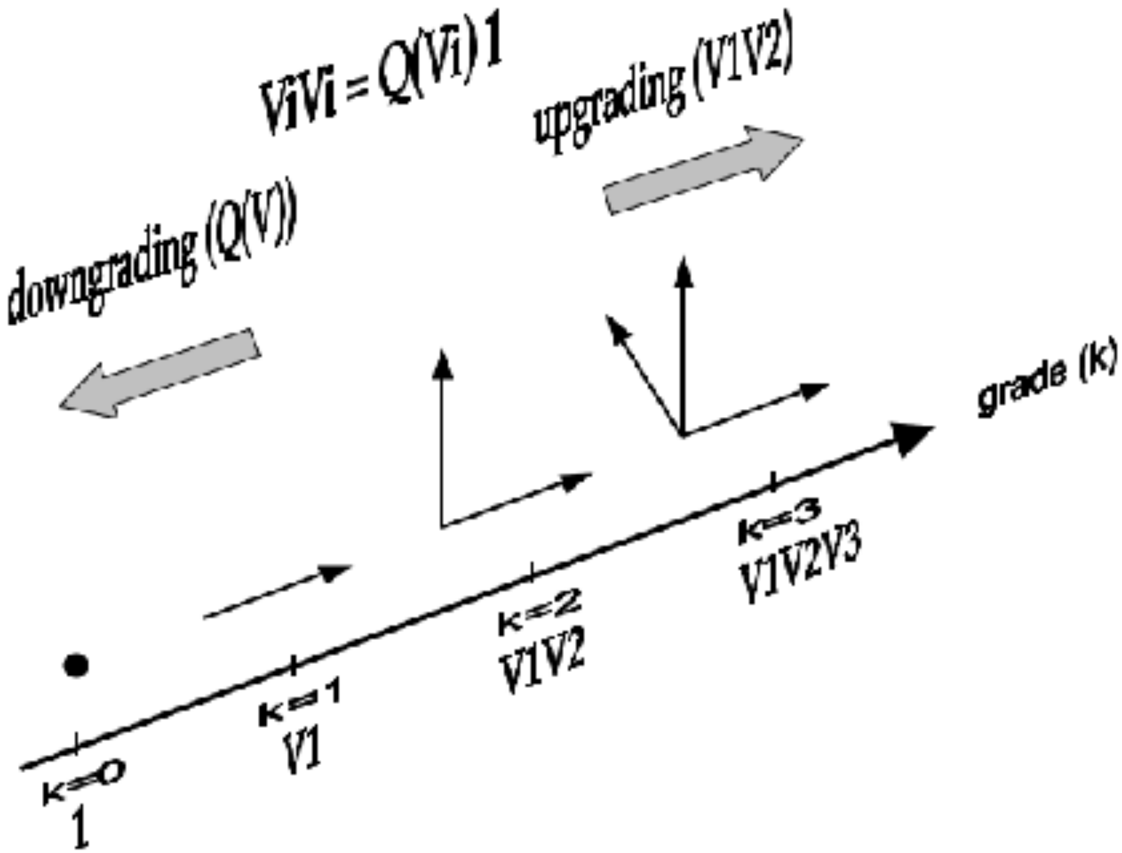
- it is generated by a n-dimensional vectorial space V={v} of operators and a quadratic form Q, of signature (p,q, $p+q=n$), which can be put into the canonical form:

$$v^2 = Q(v)1 \quad Q(v) = v_1^2 + v_2^2 .. + v_p^2 - v_{p+1}^2 - v_{p+2}^2 .. - v_{p+q}^2$$

ex.: $R = Cl_{0,0}$; $C = Cl_{0,1}$; $H = Cl_{0,2}$

H'= $l(2, \mathrm{R}) = Cl_{2,0} = Cl_{1,1}$ "pseudo-/split- quaternions"

Clifford algebra



Clifford algebra are

- graded algebra (see figure)
- double algebra:
  - 2 multiplications
- super algebra (!):

$$Cl(V,Q) = Cl^0(V,Q) \oplus Cl^1(V,Q)$$

for real algebra:

$$Cl^0_{p,q}(R) \cong Cl_{p,q-1}(R) \text{ for } q > 0$$

$$Cl^0_{p,q}(R) \cong Cl_{q,p-1}(R) \text{ for } p > 0$$

==> $R \subset C \subset H \subset O \quad \ldots$