Multifractals, Intermittency, Spectra and Climate Variability Across Scales

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A range of a billion: 0.1mm – 10,000km
Range in time scales:
a billion billion:
0.001s – 4.5 billion years

In this plot: range of scales \( >10^{17} \)
How to understand variability over range of billions (space), billion billions (time)?

Answer 1: (Still) dominant Scalebound view: Ideological and Qualitative
"Educated Guess" Mitchell’s (1976)

Murray Mitchell:

- Quasi-periodic processes dominant
- Uninteresting = white noise background
“Artist’s Rendering” (2002-2020)
“Mental Model” 2015

"Inspired by Mitchell":
National Oceanographic and Atmospheric Administration (NOAA) website 2015
Relative spectrum after “removal of background” by dividing by $1/\omega$ spectrum
The scalebound view over time

Educated Guess (1976)

Artist’s Rendition (2002, 2020)

Mental Model (2015)

Conceptual Landscape (2021)

Background: not relevant/important, ignored
How to understand variability over range of billions (space), billion billion (time)?

Answer 2:
Scaling view:
Taking the data (and background) seriously

Quantitative
The impact of data

Scaling of spectra
2000-2010: Planetary scale Horizontal Scaling

\[ E(k) = k^{-\beta} \]

Satellite: TRMM visible, IR, 1000 orbits

Satellite: TRMM microwave, 1000 orbits

Aircraft: 24 legs

Reanalysis: ECMWF interim 700 mb, ±45°, 1 year

2016: Earth Versus Mars

Comparing spectra from reanalyses

Earth

Mars

"solar system" universality
1976 versus 2015: Educated Guess versus data

\[ E(\omega) \approx \omega^{-\beta} \]

\[ \beta = 1.8 \]

Murray Mitchell:
- Quasi-periodic processes dominant
- Uninteresting = white noise background

Mitchell on his head:
- Most variability in wide range scaling processes,
- Quasi-oscillatory processes superposed

L 2015: A voyage through scales, a missing quadrillion and why the climate is not what you expect, *Climate Dynamics*
The data after “background removal”

2021 versus 2015: Conceptual Landscape versus data

Scaling spectrum

Total variance is proportional to the area (years)$^{-1}$ $\omega$
Space-time “Stommel” diagrams

Scalebound view

Ocean 1963
Stommel 1963

Atmosphere 1976
Orlanski 1976
Space-time Diagrams in 2020

From Ghil, Lucarini 2020 (courtesy D. Chelton)
Space-time Diagrams: The impact of scaling

Anisotropic extension of Kolmogorov law

\[ \Delta v \approx \varepsilon^{1/3} l^{1/3} \]

Horizontal only

Horizontal extent

Energy rate density

Velocity Fluctuation

Atmosphere
\[ \varepsilon \approx 10^{-3} \text{W/kg} \]

Ocean
\[ \varepsilon \approx 10^{-8} \text{W/kg} \]

Lifetime

\[ \tau = l^{2/3} \varepsilon^{-1/3} \]

10 days: weather - macroweather transition

6 months: ocean - weather - macroweather transition

Largest scale on earth (20,000km)

10 days: weather

6 months: ocean

Slope 2/3

Updated from Ghil, Lucarini 2020 (courtesy D. Chelton)
2012: Impact of data and scaling:
1400 Geostationary IR images ((x, y, t): 1000x1000x1400 pixels)

- Perfect scaling (with finite size effects)
- Diurnal peak
- -1.5 reference slope

$P(\lambda^{-1}(k, \omega)) = \lambda^s P((k, \omega))$  
Accurate space-time scaling

$P(\omega, k) \propto \langle |\tilde{I}(\omega, k)|^2 \rangle$
Fluctuations, Wavelets

\[ \langle \Delta T(\Delta t) \rangle \approx \Delta t^H \]

Mean fluctuation

Fluctuation (defined by wavelets...)

Fluctuation exponent
New simple technique (re)discovered in 2012: Fluctuation analysis

Scaling regimes

Based on Haar wavelets (1910)

Lovejoy 2013
Multifractality 1:

Intermittency

Spikes
Multifractality, Intermittency: Time

Veizer 553 kyr
Zachos 5,000 yr
GRIP 85 yr
ECMWF 1 month
Montreal 1 hour
Lander 1 hour
Thermistor 0.067s

All series 1000 points

ΔT / ΔT

p=10^{-6}  p=10^{-9}  p=10^{-3}

Veizer
Zachos
GRIP
ECMWF
Montreal
Lander
Thermistor

MC mC C M W W W
Multifractality, Intermittency: Space

180 points, 
(2° at 45N, 140 year average)

H > 0

360 points, 
(1° at 45N, 1 month average)

360 points, 
(1° at 45N, 1 day average)

1000 points, 
(280m, aircraft)
Fluctuations and intermittency

\[
\Delta T(\Delta t) = \phi_{\Delta t} \Delta t^H
\]

\[
\left\langle \Delta T(\Delta t)^q \right\rangle = \left\langle \phi_{\lambda}^q \right\rangle \Delta t^{qH} \propto \Delta t^{-(K(q)+qH)}
\]

\[
\xi(q) = qH - K(q)
\]

\[
\left\langle \phi_{\lambda}^q \right\rangle = \lambda^{K(q)}; \quad \lambda = \tau / \Delta t
\]

\[
\left\langle \Delta T(\Delta t) \right\rangle / \left\langle \Delta T(\Delta t)^2 \right\rangle^{1/2} \propto \Delta t^{K(2)/2} \approx \Delta t^{C_1}
\]

Mean/RMS

=0 for Gaussian processes
Macroweather spatial and temporal intermittency

Temperature changes (K)

Low intermittency (low “spikiness”)

High intermittency (high “spikiness”)

Space: 60° N, 1990, 2° resolution, annually averaged
(Haar) Fluctuations

Space (units, degrees longitude)

\[
\left\langle \left( \Delta T \left( \Delta x \right) \right)^2 \right\rangle^{1/2}
\]

\[
\left\langle \left| \Delta T \left( \Delta x \right) \right| \right\rangle
\]

\[H_{\text{space}} = 0.45\]

Converging lines high intermittency

Parallel lines low intermittency

180° latitude

Log$_{10} \Delta t$

2 months

Log$_{10} \Delta x$

4° latitude

H$_{\text{time}} = -0.20$

100 yrs

Anthropogenic warming

top are spatial temperature fluctuations, bottom temporal at equator
Multifractal interpretation

\[ \Pr(\gamma' > \gamma) \approx \lambda^{-c(\gamma)} \]

\[ \gamma_{\text{max}} = \frac{\text{Log}|\Delta T|/|\Delta T|}{\text{Log}\lambda} \]

\[ c(\gamma_{\text{max}}) = 1 - d(\gamma_{\text{max}}) = 1 - 0 = 1 \]

\[ \text{Theory: } \gamma_{\text{max}} = 0.43 \]

\[ d(\gamma_1) = 1 - c(\gamma_1) \]

\[ d(\gamma_2) = 1 - c(\gamma_2) \]

\[ p_{\text{Gauss}} = 10^{-6} \]

\[ \lambda = \text{largest/smallest} = 360 \]

Gradient

\[ \frac{|\Delta T|}{|\Delta T|} = \lambda^\gamma \]

\[ |\Delta T| = \lambda^\gamma \]

(weather, space)
Multifractality 2: Cascades
Cascades

Generic statistical behaviour:

\[
\langle \mathcal{E}^q \rangle \approx \lambda^{K(q)}
\]

Scale invariant

Statistical averaging

Resolution: ratio \( \lambda = L/l \)

\( \lambda \)

\( K(q) \)

\( q \)

\( \mathcal{E} \)

\( L \)

\( l \)

\( S+L \ 1987 \)

= multifractal
Intermittency, multifractality (sparse, “spikes”)

.....the exponents $C_1, \alpha$

Statistics of the “spikes”

$$\frac{|\Delta T|}{|\Delta T|} = \lambda^\gamma$$

$$\left\langle \varepsilon_{\lambda}^q \right\rangle = \lambda^{K(q)}; \quad \text{Pr}(\varepsilon > \lambda^\gamma) \approx \lambda^{-c(\gamma)}$$

$$c(\gamma) = \max_q (q\gamma - K(q))$$

Legendre transformation
(Parisi and Frisch 1985)

$$K(q) = \max_\gamma (q\gamma - c(\gamma))$$

Characterization near the mean:

$$C_1 = K'(1)$$

$\alpha$ and Universal multifractals
(Parisi and Frisch 1985)

$$K(q) = \frac{C_1}{\alpha - 1} (q^\alpha - q); \quad c(\gamma) = C_1 \left( \frac{\gamma}{C_1 \alpha'} + \frac{1}{\alpha} \right)^{\alpha'}$$

$$\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$$

$$0 \leq \alpha \leq 2$$
Empirical analysis: Estimating fluxes from the fluctuations

Multifractal cascade equation:

\[ \langle \varphi^q \rangle = \lambda^{K(q)} \]

\[ \Delta T = \varphi_{\Delta t} \Delta t^H \]

Fluctuations:

Estimating the fluxes from the fluctuations

\[ \varphi' = \frac{\varphi_\lambda}{\langle \varphi_\lambda \rangle} \approx \frac{\Delta T (\Delta t)}{\langle \Delta T (\Delta t) \rangle}; \quad \lambda = \frac{\tau}{\Delta t} \]

Normalized flux at resolution \( \lambda \)

The “spikes”

\[ M_q = \langle \varphi'^q \rangle \]

“Trace moments”

= The statistics of the spikes at different scales

Estimate at finest resolution, then degrade to intermediate resolutions by averaging

outer cascade scale

"Trace moments"
Early evidence of cascades:
Precipitation 1987

(70 Radar Scans, Montreal, horizontal 3 weeks of rain data)

Cascade prediction:

\[
\frac{\langle Z^q \rangle}{\langle Z \rangle^q} = \lambda^{K(q)}
\]

\[
\lambda = \frac{L_{\text{eff}}}{L_{\text{res}}}
\]

\[
M = \frac{\langle Z^q \rangle}{\langle Z \rangle^q}
\]

Schertzer and Lovejoy 1987
Tropical Rainfall Measuring Mission

TRMM: $10^9$ over 10 years
Scale-dependent TRMM PR Attenuation Corrected Reflectivity Factor $[\lambda_Z]$ (1176 consecutive orbits -- ~70 days)

$Z_\lambda \equiv$ scale-dependent, attenuation corrected, reflectivity factor over 250 m thick layer just above surface

$M_\lambda = \frac{\langle Z_\lambda^q \rangle}{\langle Z_\lambda \rangle^q} \equiv \lambda^{K(q)}$ scale-dependent normalized moment

$\lambda = \frac{L_{Earth} (20000 \text{ km} \approx \pi r_e)}{L_{res}} \equiv$ normalized distance scale

$q \equiv$ fractional moment

Lovejoy et al. 2008

residual variability at Earth size

20,000 km

1000 km

100 km

4.3 km

Spurious curvature of the support (q=0)
$M_q \approx \lambda^{K(q)}$

Cascades horizontal

20CR 45°N, zonal

EW direction

EW wind

NS wind

Temp

humidity
\[ M_q = \left\langle \epsilon^q \right\rangle \]

Flux resolution \( \lambda \)

\[ M_q \approx \lambda^K(q) \]

Predictions of cascade models

Chen, Lovejoy and Muller 2016
### Earth versus Mars:

nearly identical multifractal cascades

<table>
<thead>
<tr>
<th></th>
<th>Mars at 83% of Surface Pressure</th>
<th>Earth at 69% of Surface Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>U</td>
<td>V</td>
</tr>
<tr>
<td>(C_1)</td>
<td>0.078</td>
<td>0.075</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>1.97</td>
<td>1.98</td>
</tr>
<tr>
<td>(L_{eff}) Ratio</td>
<td>0.42</td>
<td>0.51</td>
</tr>
</tbody>
</table>

U = Zonal Wind, V = Meridional Wind, T = Temperature

Data for Earth are from the ECMWF Reanalyses: Lovejoy and Schertzer (2011)
Energy budget

TRMM satellite data, ≈1000 orbits

Energy source
Visible
- q=2
- q=1.6
- q=1.2
- q<1

Energy sink
thermal IR
- 20000 km
- 10 km
Horizontal cascades from 24 aircraft legs (11-13km)

Fields that are relatively unaffected by the trajectories
Governing Equations and Numerical Weather Prediction Models
Governing atmospheric Equations

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} & = -(\mathbf{u} \cdot \text{grad})\mathbf{u} - 2\Omega \times \mathbf{u} - \alpha \text{grad} p - \text{grad} \Phi + \mathbf{F} \\
\frac{\partial T}{\partial t} & = -c_v(\mathbf{u} \cdot \text{grad})T - \frac{p}{\rho} \text{div} \mathbf{u} + Q \\
\frac{\partial \rho}{\partial t} & = -(\mathbf{u} \cdot \text{grad})\rho - \rho \text{div} \mathbf{u} \\
p & = \rho R T
\end{align*}
\]

Important property: Scaling symmetry

Atmospheric laws \quad \text{Anisotropic blowup} \quad \lambda^H \quad \text{(Atmospheric laws)}

Factor \lambda

Conservation of momentum
Conservation of energy
Conservation of matter
Equation of state
Global GEMS Model 00h (Numerical Weather Prediction Model)

Analysis of four months U, T at 1000 mb

(48 h forecasts are almost the same)
Summary
Horizontal spatial Scaling exponents

\[ \Delta I = \varphi \Delta x^H \]
\[ \langle \varphi_x^q \rangle = \lambda^{K(q)} \]
\[ \lambda = \frac{C_1}{\alpha - 1} \frac{q^\alpha - q}{\Delta x} \]
\[ E(k) \approx k^{-\beta} \]

L+S 2013

**State variables**
- \( u, v \)
- \( w \)
- \( T \)
- \( h \)
- \( z \)

<table>
<thead>
<tr>
<th>State variables</th>
<th>( C_1 )</th>
<th>( \alpha )</th>
<th>( H )</th>
<th>( \beta )</th>
<th>( L_{\text{eff}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u, v )</td>
<td>0.09</td>
<td>1.9</td>
<td>1/3, 0.77</td>
<td>1.6, 2.4</td>
<td>14 000</td>
</tr>
<tr>
<td>( w )</td>
<td>(0.12)</td>
<td>(1.9)</td>
<td>(−0.14)</td>
<td>(0.4)</td>
<td>(15 000)</td>
</tr>
<tr>
<td>( T )</td>
<td>0.11, (0.08)</td>
<td>1.8</td>
<td>0.50, 0.77</td>
<td>1.9, 2.4</td>
<td>5000 (19 000)</td>
</tr>
<tr>
<td>( h )</td>
<td>0.09</td>
<td>1.8</td>
<td>0.51</td>
<td>1.9</td>
<td>10 000</td>
</tr>
<tr>
<td>( z )</td>
<td>(0.09)</td>
<td>(1.9)</td>
<td>(1.26)</td>
<td>(3.3)</td>
<td>(60 000)</td>
</tr>
</tbody>
</table>

**Precipitation**
- \( R \)

<table>
<thead>
<tr>
<th>Precipitation</th>
<th>( \varphi )</th>
<th>( \lambda )</th>
<th>( L_{\text{eff}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>0.4</td>
<td>1.5</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**Passive scalars**
- Aerosol concentration

<table>
<thead>
<tr>
<th>Passive scalars</th>
<th>( C_1 )</th>
<th>( \alpha )</th>
<th>( H )</th>
<th>( \beta )</th>
<th>( L_{\text{eff}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aerosol concentration</td>
<td>0.08</td>
<td>1.8</td>
<td>0.33</td>
<td>1.6</td>
<td>25 000</td>
</tr>
</tbody>
</table>

**Radiances**
- Infrared
- Visible
- Passive microwave

<table>
<thead>
<tr>
<th>Radiances</th>
<th>( C_1 )</th>
<th>( \alpha )</th>
<th>( H )</th>
<th>( \beta )</th>
<th>( L_{\text{eff}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infrared</td>
<td>0.08</td>
<td>1.5</td>
<td>0.3</td>
<td>1.5</td>
<td>15 000</td>
</tr>
<tr>
<td>Visible</td>
<td>0.08</td>
<td>1.5</td>
<td>0.2</td>
<td>1.5</td>
<td>10 000</td>
</tr>
<tr>
<td>Passive microwave</td>
<td>0.1–0.26</td>
<td>1.5</td>
<td>0.25–0.5</td>
<td>1.3–1.6</td>
<td>5000–15 000</td>
</tr>
</tbody>
</table>

**Topography**
- Altitude

<table>
<thead>
<tr>
<th>Topography</th>
<th>( C_1 )</th>
<th>( \alpha )</th>
<th>( H )</th>
<th>( \beta )</th>
<th>( L_{\text{eff}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Altitude</td>
<td>0.12</td>
<td>1.8</td>
<td>0.7</td>
<td>2.1</td>
<td>20 000</td>
</tr>
</tbody>
</table>

**Sea surface temperature**
- SST (see Table 8.2)

<table>
<thead>
<tr>
<th>Sea surface temperature</th>
<th>( C_1 )</th>
<th>( \alpha )</th>
<th>( H )</th>
<th>( \beta )</th>
<th>( L_{\text{eff}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SST (see Table 8.2)</td>
<td>0.12</td>
<td>1.9</td>
<td>0.50</td>
<td>1.8</td>
<td>16 000</td>
</tr>
</tbody>
</table>

**Summary**
- \( \alpha \): 1.5 – 1.9
- \( C_1 \): range 0.08 - 0.12... except precipitation!
- \( L_{\text{eff}} \): outer scale \( \approx 20,000\)km
Empirical Conclusions of planetary scale horizontal analyses

1) Multifractal scaling (including spectra) with outer scale close to the scale of the planet is respected to within ±1% to ±2% for moments up to order q=2, up to 5000km.

2) It is accurately respected by
   a) remotely sensed radiances
   b) Reanalyses
   c) Aircraft data
   d) Numerical weather prediction models
   e) On Mars: (same multifractal exponents)

How is this possible?!
Which symmetry is primary: isotropy or scaling?

Isotropy first – then scaling

Motivation:
- the simplicity of isotropic theory
- theoretical approximations that confine the effects of gravity (buoyancy) to small scales

2-D isotropic
("Quasi-geostrophic") turbulence

$D_{el}=2$

Only Scaling (anisotropic)

Motivation:
- Scaling of the governing equations
- gravity acts at all scales
- numerical models, data

3D isotropic turbulence

$D_{el}=3$

$\approx 10$km
Anisotropy: the vertical

Velocity structure functions 237 drop sondes

No Kolmogorov 3D isotropic turbulence above 5m!

Log$_{10}\langle|\Delta v_x|\rangle$

Log$_{10}|u(z+\Delta z)-u(z)|$

$H=1$: Constant Brunt-Vaisala frequency, quasi-linear gravity waves, or pseudo potential vorticity

$H=3/5$: Bolgiano-Obukhov value

$H=1/3$: Kolmogorov, 3D isotropic turbulence

Lovejoy, S., S. Hovde, A. Tuck, D. Schertzer, 2007
Vertical cascades:
Thermodynamic fields (Dropsonde data)

\[ M = \left\langle \phi_\lambda^q \right\rangle / \left\langle \phi^q \right\rangle \]
\[ M_q \approx \lambda^{K(q)} \]
Vertical cascades: lidar backscatter

From 10 airborne lidar cross-sections near Vancouver B.C.

Horizontal cascade

Vertical cascade

\[ M = \frac{\langle \delta I^q \rangle}{\langle \delta I^1 \rangle^q} \]

- \( M = \frac{C_1}{\lambda^q} \)
  - \( C_1 = 0.076 \)
  - \( q = 1, 1.6, 2 \)

- \( M = \frac{C_1}{\lambda^q} \)
  - \( C_1 = 0.11 \)
  - \( q = 1, 1.6, 2 \)

L, Tuck, Hovde, S, 2009
The physical scale function and differential scaling

\[ |\Delta r| \rightarrow \|\Delta r\| \]

Usual distance (=vector norm)  
Scale function (scale notion)

Scale symmetry  
\[ \|\lambda^{-G} x\| = \lambda^{-1} \|x\| \]

"canonical" scale function:  
\[ \|(\Delta x, \Delta z)\| = l_s \left( \left( \frac{\Delta x}{l_s} \right)^2 + \left( \frac{\Delta z}{l_s} \right)^{2/H_z} \right)^{1/2} \]

\[ G = \begin{pmatrix} 1 & 0 \\ 0 & H_z \end{pmatrix} \]

Isotropic function  
\[ H_z = 1 \]

Vertical sections

Anisotropic physical scale function  
\[ H_z = 5/9 \]

Bolgiano-Obhukhov

Kolmogorov

Sphero-scale

S+L 1985
The 23/9D model

\[ D_{el} = 2 \]

\[ \Delta v \Delta x \approx \varepsilon \frac{1}{3} \Delta x^{\frac{1}{3}} \]

\[ \Delta v \Delta z \approx \phi \frac{1}{5} \Delta z^{\frac{3}{5}} \]

\[ H_z = \frac{1}{3} / \left( \frac{3}{5} \right) = \frac{5}{9} \]

The 23/9D dynamics:

\[ \text{Kolmogorov} \]

\[ \text{Bolgiano-Obukhov} \]
Anisotropic, Stratified Scaling

Stochastic

5km

Blow up X 2.9 (each)

Isotropic

Total: X5000

1 m

$H_z = 1$

$H_z = \frac{5}{9}$
14500 aircraft flights: 5-5.5km altitude, 2009, US (TAMDAR data)

$$\langle |\Delta v(\Delta x, \Delta z)|^2 \rangle \ (\text{m/s})^2$$

Purple = theory
Black = measurements

Longitudinal

Transverse

Velocity structure function
$$\langle \Delta v^2 (\Delta x, \Delta z) \rangle = C \| (\Delta x, \Delta z) \|^{\xi(2)}$$
$$\xi(2) \approx 0.80$$

Canonical scale function
$$\| (\Delta x, \Delta z) \| = \left( \left( \frac{\Delta x}{l_s} \right)^2 + \left( \frac{\Delta z}{l_s} \right)^{2/H_z} \right)^{1/2}$$
$$H_z \approx 0.57 \pm 0.01$$

(Theory: $5/9=0.555...$)

Pinel L+S, 2012
Empirical estimates of $H_z$:

Aircraft compared to drop sondes

<table>
<thead>
<tr>
<th>T</th>
<th>Log$\theta$</th>
<th>h</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.47±0.09</td>
<td>0.47±0.09</td>
<td>0.65±0.06</td>
<td>0.46±0.05</td>
</tr>
</tbody>
</table>

L+S, 2013

14500 aircraft trajectories: $H_z = 0.57±0.01$ (Pinel, L+S 2012)

Lidar aerosol cross sections: $H_z \approx 0.53±0.02$ (Lilley, L+S, Strawbridge, 2008)

16 Clousat orbits, radar reflectivity: $H_z = 0.56±0.04$ (L+S, Tuck 2009)

Historical development of GCM’s: $H_z \approx 5/9$ (L 2019)

$\approx 10^6$ CloudSat cloud heights, thicknesses: $H_z = 0.53±0.02$ (L 2021)

Conclusion:
The $D_{el} = 2+Hz=23/9= 2.55$ model is well supported by diverse data.
New and old results on the divergence of moments at high Re turbulence

Theoretical prediction of multifractal processes

\[
\Pr(\Delta v > s) \approx s^{-q_D}
\]

Large threshold \( s \)

\[ \langle \Delta v^q \rangle \to \infty; \quad q > q_D \]

\[
K(q_D) = D(q_D - 1)
\]

Multifractal theory
(Mandelbrot 1974, Kahane and Peyriere 1976, S+L 1983)
Wind (Vertical direction)

Horizontal wind in the, vertical From Radiosondes

\[
\Pr\left(\Delta v > s\right) \approx s^{-q_{D,v}}
\]

\[q_{D,v} = 3q_{D,\varepsilon} \approx 5\]

\[
\varepsilon = \Delta v^3 / \Delta x
\]

Wind (Horizontal direction)

Horizontal wind (aircraft data)

\[
\Pr\left(\varepsilon > s\right) \approx s^{-q_{D,\varepsilon}}
\]

S+L 1985
Wind fluctuations, time, 10 Hz, (sonic anemometer)

Schmitt, S+ L, Brunet, 1994

Aircraft data at 40, 80 m separation

L+S, 2007
Wind tunnel turbulence

\[ \Pr \left( \frac{\Delta v}{v_{RMS}} > s \right) \]

Theory

- DR: \( \varepsilon \propto \Delta v^2 \)
- IR: \( \varepsilon \propto \Delta v^3 \)

\[ \frac{q_{D,v,DR}}{q_{D,v,IR}} = \frac{3}{2} \]

Empirical: \( 7.7/5.4 \approx 1.43 \)

Inertial range (IR)

- \( q_{D,v,IR} \approx 7.7 \)

Dissipation range (DR)

- \( q_{D,v,DR} \approx 5.4 \)

[100x80]2013
**Conclusion of all studies:** $q_{Dv} \approx 5$
Generalized Scale Invariance
Scale functions in linear GSI (position independent)

Isotropic (self similar)

\[ T_\lambda = \lambda^{-G} \]

\[
G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Scale functions

\[
\left| \lambda^{-G} \mathbf{r} \right| = \lambda^{-1} |\mathbf{r}| 
\]

Stratification dominant (real eigenvalues)

\[
G = \begin{pmatrix} 1.35 & 0.25 \\ 0.25 & 0.65 \end{pmatrix}
\]

Self-affine

Scale isolines in red \( |\mathbf{r}| \) = constant

Rotation dominant (complex eigenvalues)

\[
G = \begin{pmatrix} 1.35 & 0 \\ 0 & 0.65 \end{pmatrix}
\]

\[
G = \begin{pmatrix} 1.35 & -0.45 \\ 0.85 & 0.65 \end{pmatrix}
\]

S+L 1985
Changing $G$


$$G = \begin{pmatrix} 1 - i & -j \\ j & 1 + i \end{pmatrix}$$
Fly by of anisotropic (multifractal, cascade) cloud
Cascades from localized to increasingly unlocalized structures:

$H_{\text{wav}} = 1/3 - H_{\text{tur}}$
Conclusions

1) Temporal Scaling defines the five main dynamical dynamical regimes: weather, macroweather, climate, macroclimate, megaclimate.

2) Scaling allows for a quantitative understanding of the atmosphere whereas usual scale bound approaches are qualitative and do not agree with the data (error $10^{15}$).

3) Origin of scaling is the (anisotropic) scaling of the governing equations (including of wind and boundary conditions).

4) Multifractals are the generic scaling process: atmospheric fields show excellent cascades structures up to planetary scales (horizontal) and $\approx 10$km (vertical).

5) This is possible due to anisotropic scaling (Generalized Scale Invariance).

6) Extreme events/ divergence of moments as theoretically predicted (order $\approx 5$ for velocity).

7) Realistic space-time simulations are possible.

References: