Global vs local multifractal

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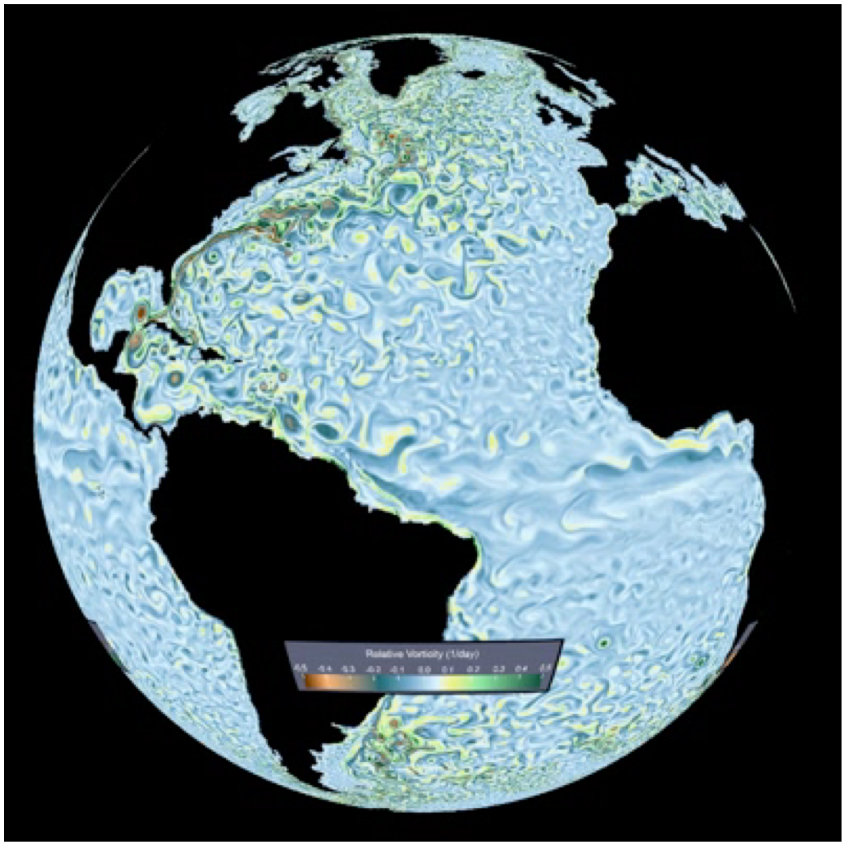
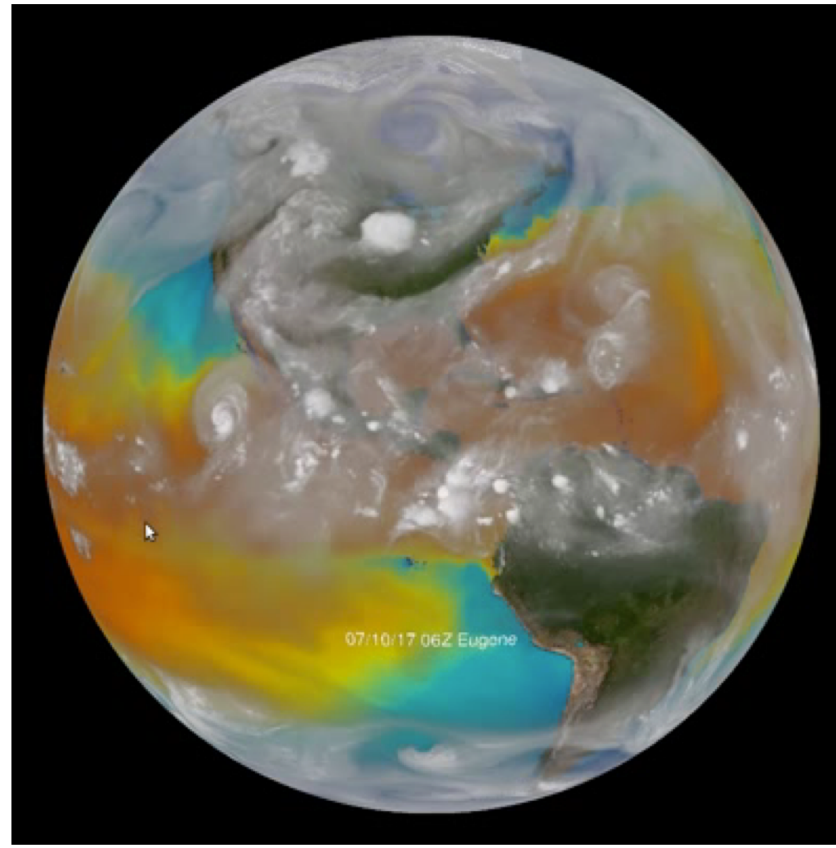
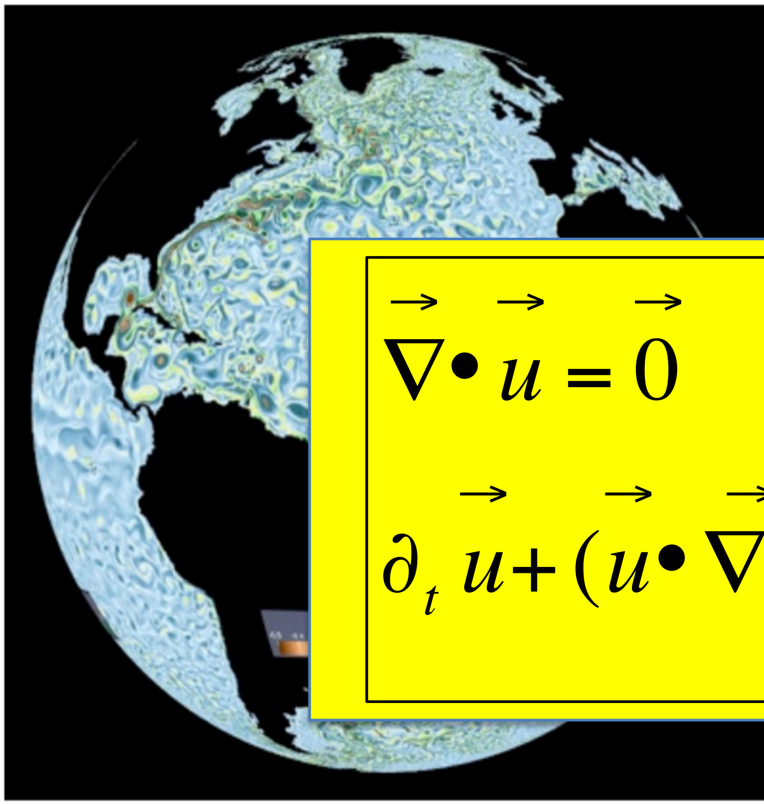
CNRS UMR 3680

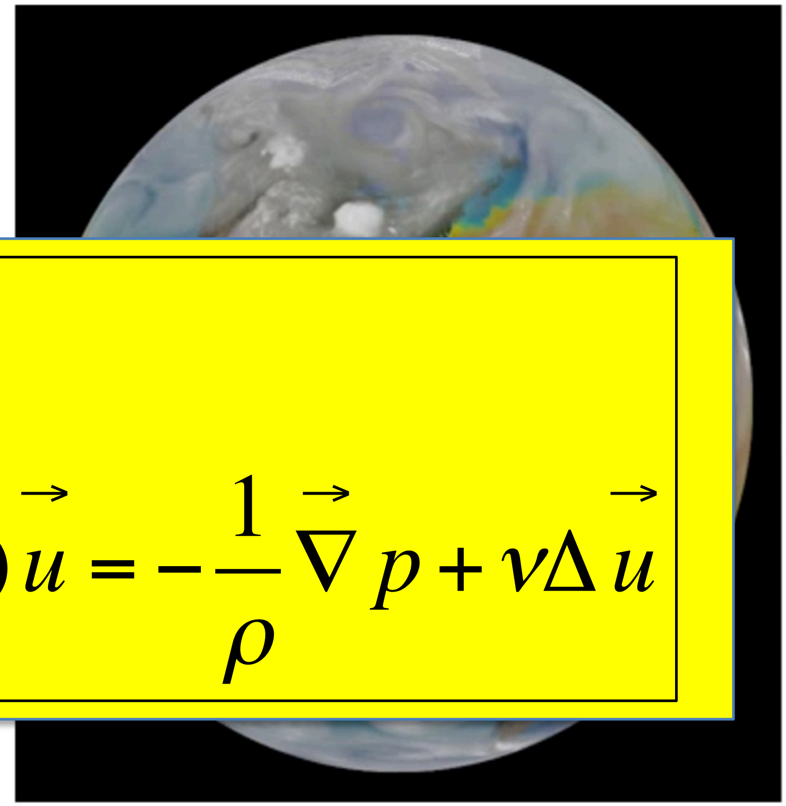
Fluids and Vortices around us





The model





$$\vec{\nabla} \bullet \vec{u} = \vec{0}$$

$$\partial_t \vec{u} + (\vec{u} \bullet \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla} p + \nu \Delta \vec{u}$$

Self-similarity of vortices

Seymour Narrows,
Between Vancouver and Quadra Islands

APRIL 6TH SHIPS SAILING SAFELY THROUGH SEYMOUR NARROWS

1962: 1st Observation

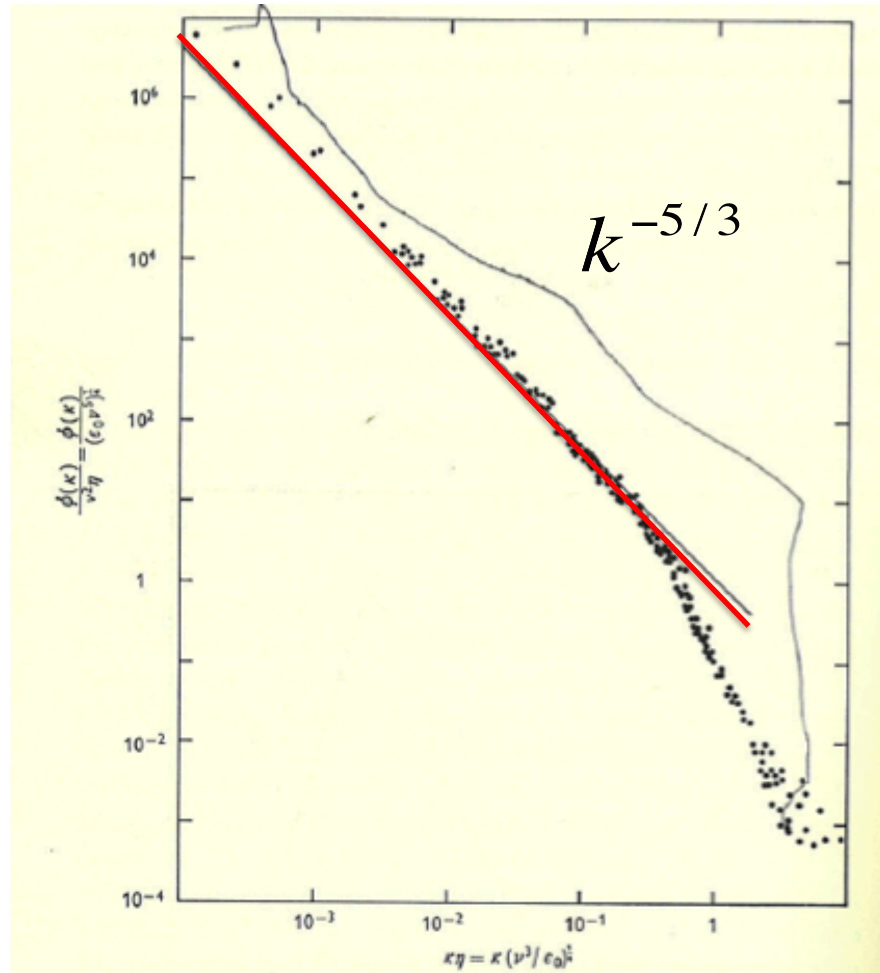


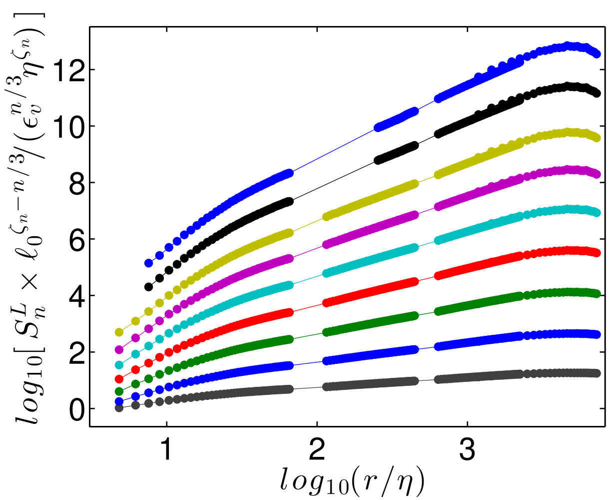
Fig. 6.2. The turbulence spectra, measured by Grant, Stewart and Moilliet (1962) and scaled according to the Kolmogorov parameters. The viscous dissipation rate ϵ_0 varied over a range of values of the order 100. The straight line represents variation as $\kappa^{-\frac{5}{3}}$. The top few points are believed to be rather high on account of the low frequency heaving motions of the ship.

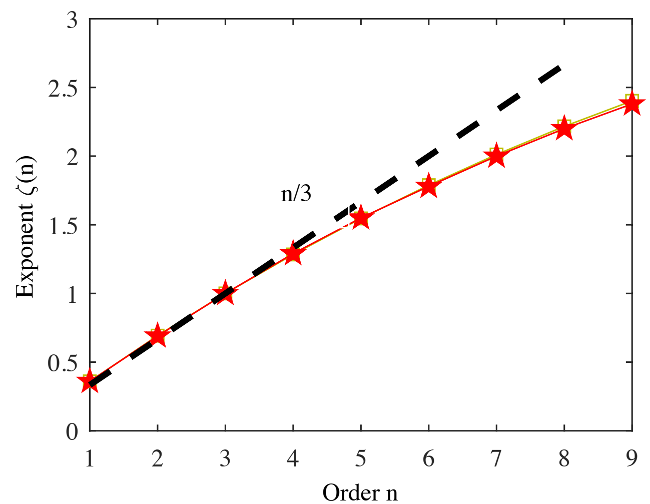
Breaking of the 1/3-scaling symmetry

Broken

$$(t,\vec{x},\vec{u}) \rightarrow (\lambda^{1-h}t, \lambda\vec{x}, \lambda^{h}\vec{u}) \quad \nu = 0$$

h=1/3 is not true?



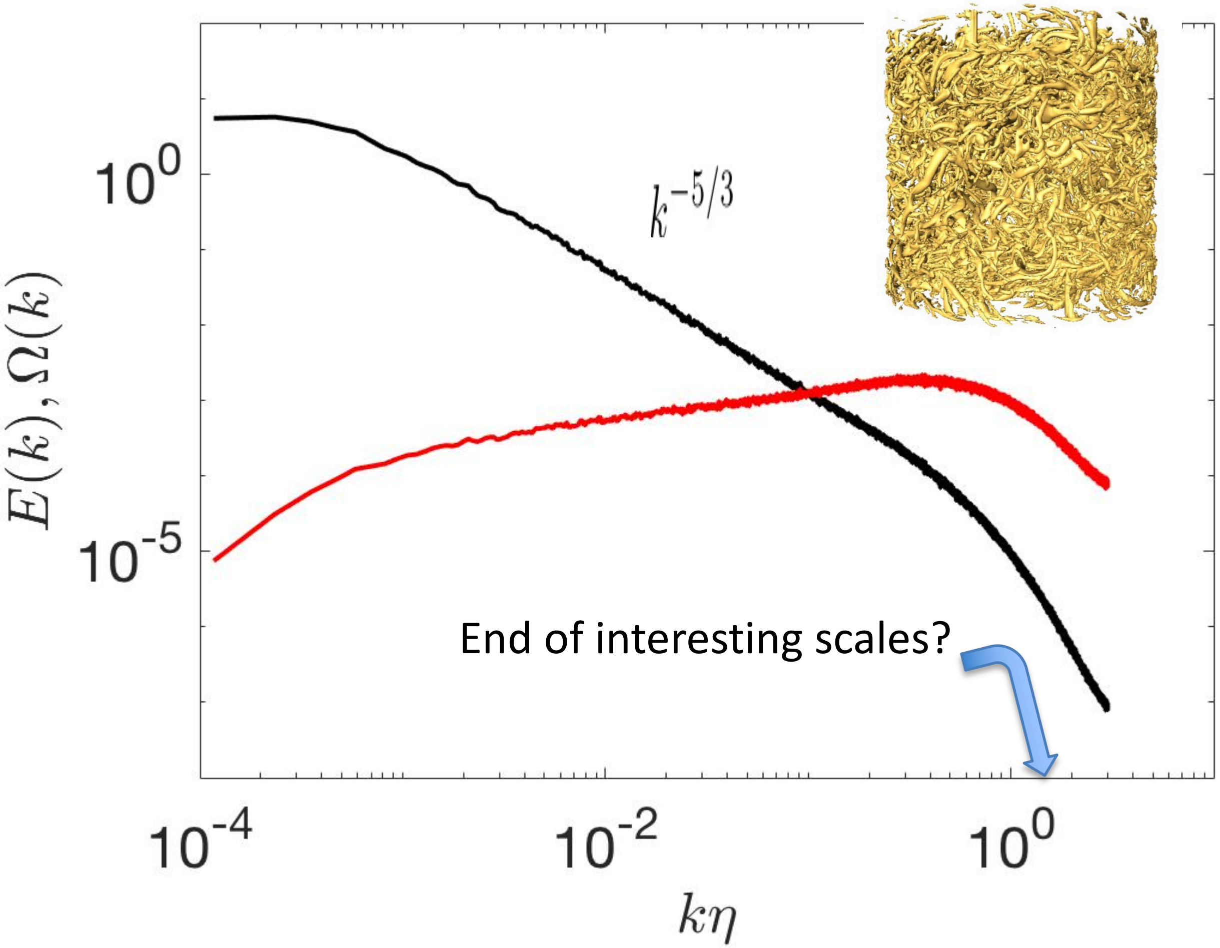


There is a problem with K41!

$$\delta u \sim \ell^{1/3} \Longrightarrow \quad S_p(\ell) =< (\delta u)^p > \sim \ell^{p/3}$$

Not observed!

The rough nature of the velocity field



While **kinetic energy** is small below η **enstrophy** is not!

Velocity gradients increase as resolution scale is decreased!

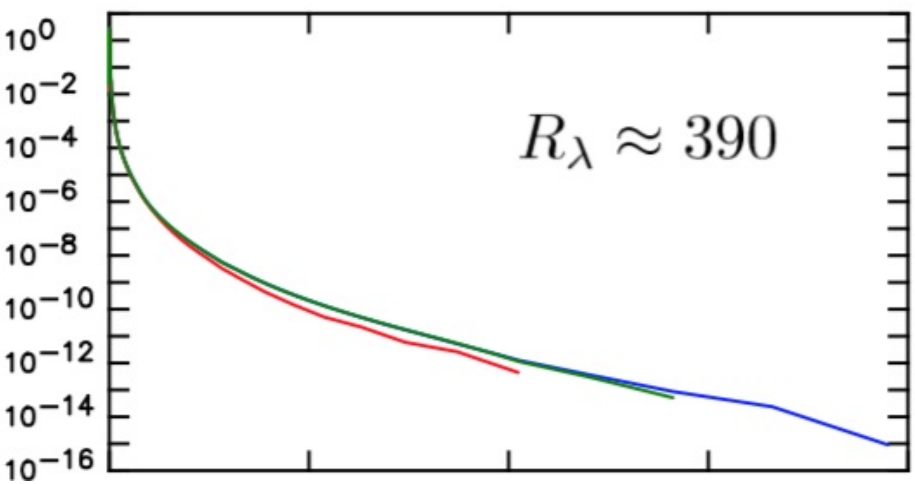
$k_{max}\eta =$ 1.4

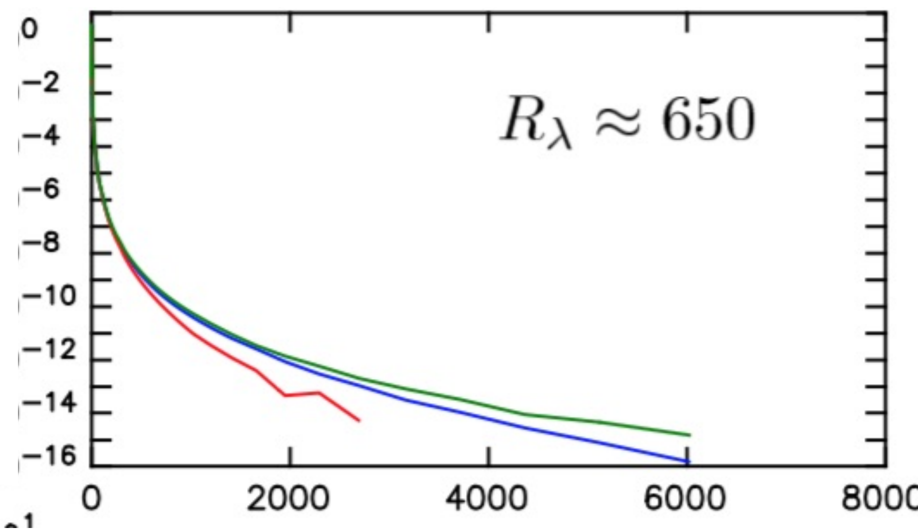
$k_{max}\eta =$ 2.8

$k_{max}\eta =$ 5.6

What happens at infinite resolution?

Gradients





Yeung et al, 2018

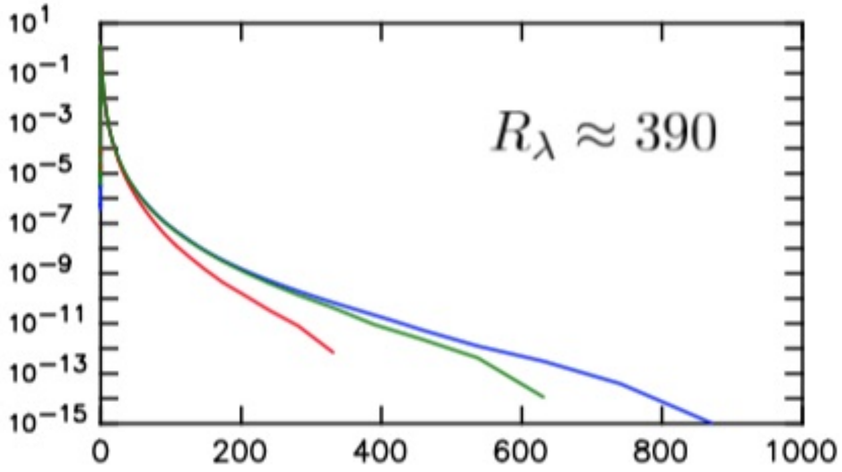
What about dissipation?

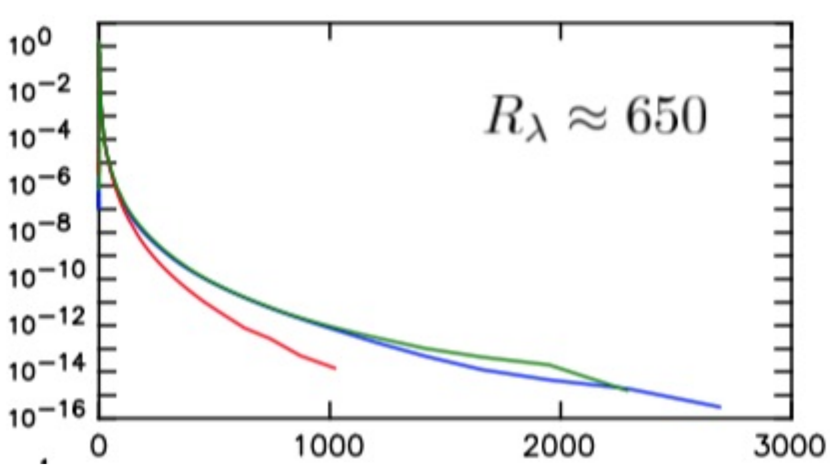
$k_{max}\eta =$ 1.4

$k_{max}\eta =$ 2.8

$k_{max}\eta =$ 5.6

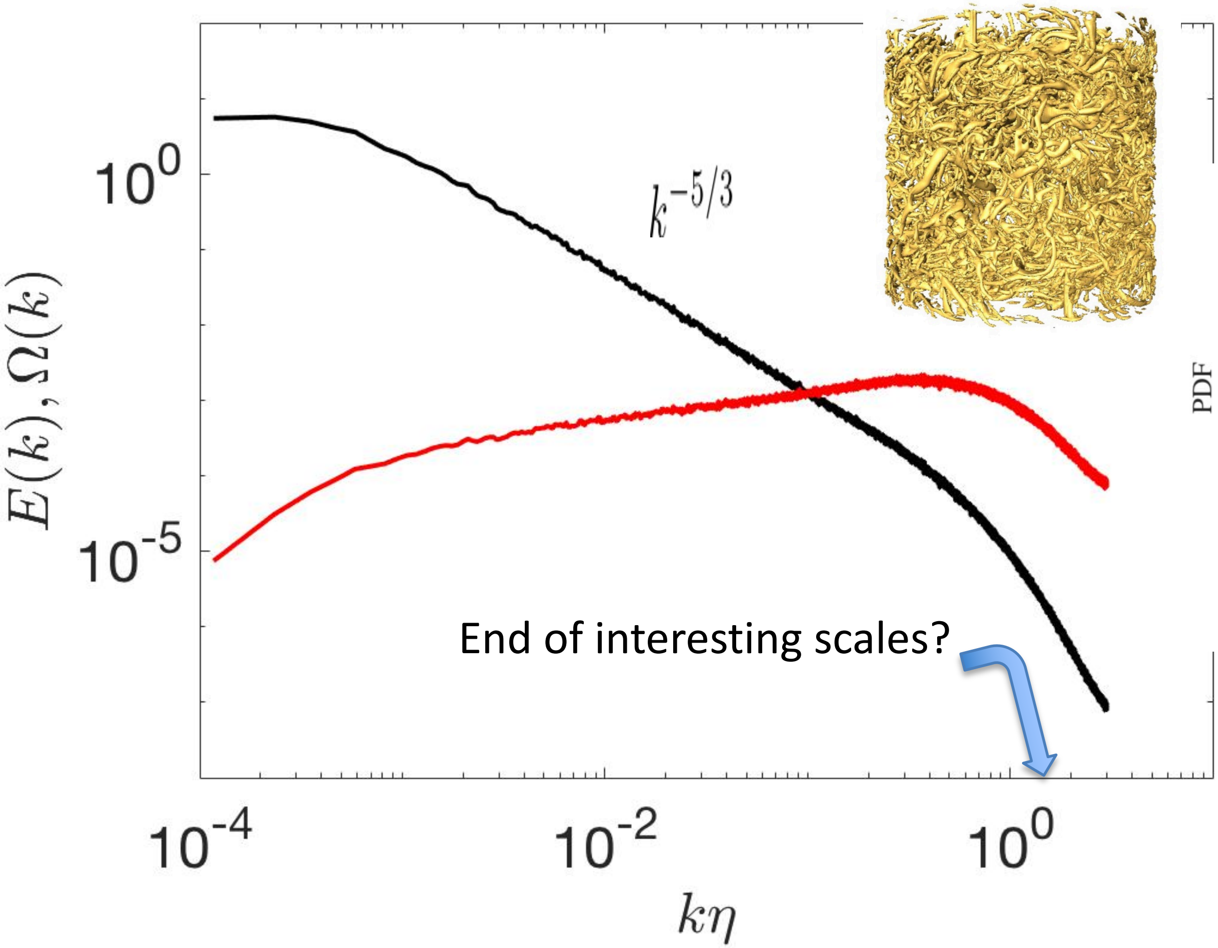
Dissipation



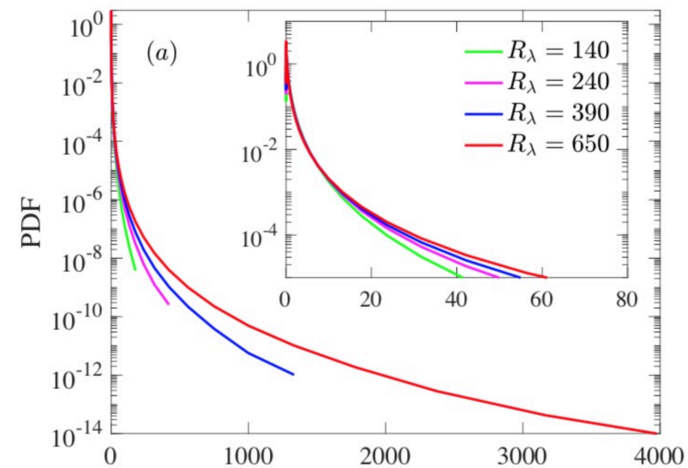


Yeung et al, 2018

The rough nature of the velocity field



End of interesting scales?



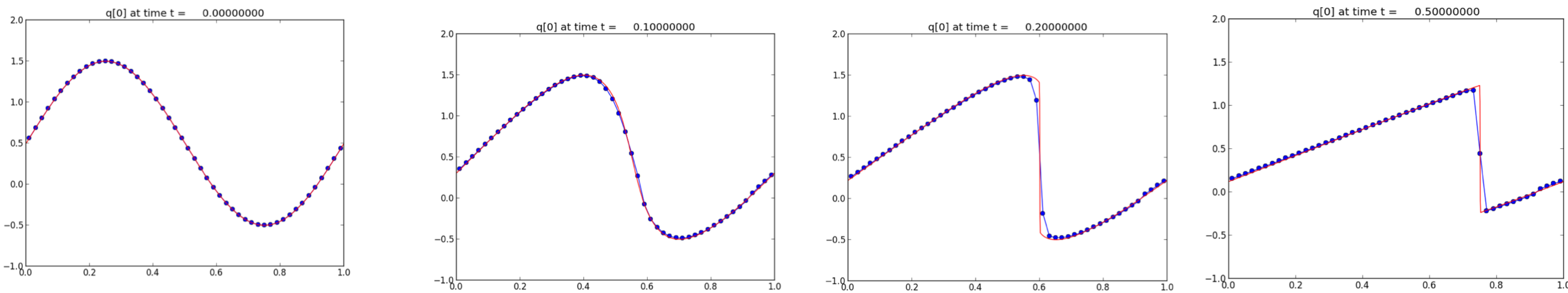
Building of extreme velocity gradients at increasing Reynolds

Bhuaria et al, 2020

How do these large velocity gradients end to?

Fate of large gradients: Burgers

$$\partial_t u + u\partial_x u = \nu\, \partial_{xx} u \qquad \nu \to 0$$



A regular intial condition…

…ends in **singularity!**

What is the situation with Navier-Stokes?

Regularity of Navier-Stokes equations

$$\vec{\nabla} \bullet \vec{u} = \vec{0}$$

$$\partial_t \vec{u} + (\vec{u} \bullet \vec{\nabla}) \vec{u} = -\frac{1}{\rho} \vec{\nabla} p + \nu \Delta \vec{u}$$

$$u(0,.) = u_0$$

Millenium Problem:
Well posedness of the Cauchy problem for finite energy solutions: existence, uniqness, regularity

2D: yes *(Ladyzhenskaya, 1958)*
3D: existence of global weak solutions. *(Leray, 1934)* but uniqness and regularity=open for

$u \in L^p(0;T;L^q(\mathbb{R}^3))$ with $\frac{2}{p} + \frac{3}{q} > 1$ (Serrin's criterium)

Singularities:Navier-Stokes vs Euler

Blow-up criteria

Singularities if diverging velocity

$$\int_0^{T_*} ||u(x,t)||_\infty^2 dt = \infty.$$

$$||u(\cdot,t)||_3 = \infty.$$

Singularities if diverging vorticity

$$\int_0^{T_*} ||\omega(x,t)||_\infty dt = \infty.$$

Existence of Self-similar Solutions

$$u(t,x) = \frac{1}{\sqrt{2\kappa t}} U\left(\frac{x}{\sqrt{2\kappa t}}\right)$$

No non-zero BSS solution ***(Tsai, 1998)***

Existence of FSS solution for

$$u_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\})$$

(Jia&Sverak, 2013)

Existence of a self-similar solution

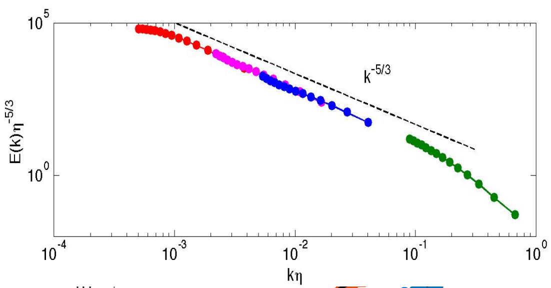
$$\omega(x,t) = \frac{1}{t-t_*} F\left(\frac{x}{(t-t_*)^\varsigma}\right)$$

Elgindi 2020

Regularity for physicist: Holder continuity

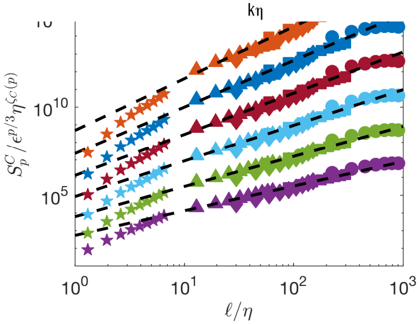
Hölder continuity: $\exists(C,\epsilon,h),\ s.t.\ \forall\ell<\epsilon,\quad |\vec{u}\,(\mathbf{x}+\ell)-\vec{u}\,(\mathbf{x})\,|<C\ell^h.$

Building Blocks: velocity increments $\delta\vec{u}(\mathbf{x},\ell)=\vec{u}(\mathbf{x}+\ell)-\vec{u}(\mathbf{x})$



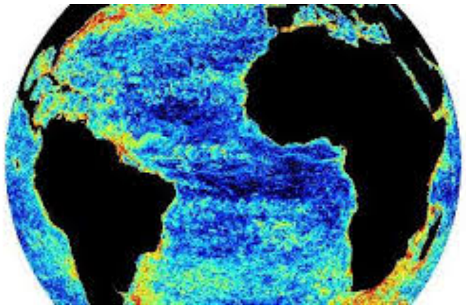
Karman-Howarth Equation:

$$\frac{1}{2}\partial_t\langle(\delta\vec{u})^2\rangle-\epsilon=\frac{1}{4}\nabla_{\mathbf{r}}\cdot\langle\delta\vec{u}(\delta\vec{u})^2\rangle-\frac{\nu}{2}\nabla^2_{\mathbf{r}}\langle(\delta\vec{u})^2\rangle$$



Velocity Structure Functions:

$$\mathcal{S}_p=\langle(\delta\vec{u})^p\rangle\sim\ell^{\zeta(p)}$$



Local energy dissipation:

$$\epsilon_\ell=\frac{1}{4}\int d\vec{\xi}\,\nabla\phi^\ell(\xi)\cdot\delta_\xi\vec{u}(\delta_\xi\vec{u})^2+\frac{\nu}{2}\int d\vec{\xi}\,\nabla^2\phi^\ell(\xi)(\delta_\xi\vec{u})^2.$$

$$\sim_{\ell\to 0}-D(\vec{u})-\nu(\nabla\vec{u})^2$$

Regularity for physicist: singularity vs quasi-singularity

Hölder continuity: $\exists(C,\epsilon,h),\ s.t.\ \forall\ell<\epsilon,\quad |\vec{u}(\mathbf{x}+\ell)-\vec{u}(\mathbf{x})|<C\ell^h.$

Building Blocks: velocity increments $\delta\vec{u}(\mathbf{x},\ell)=\vec{u}(\mathbf{x}+\ell)-\vec{u}(\mathbf{x})$

$$\delta\vec{u}=\vec{u}(x+\ell)-\vec{u}(x)\sim_{\ell\to 0}\ell^h$$

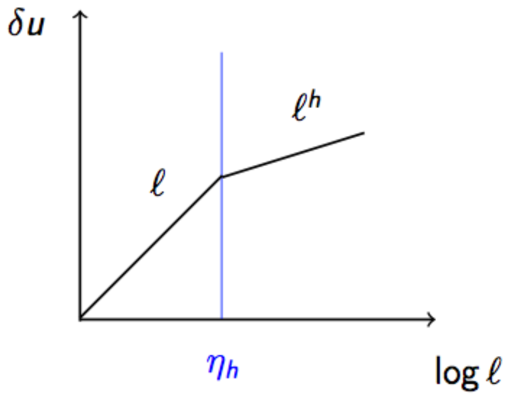
Blow-up of velocity gradients:

$$\nabla u\sim\lim_{\ell\to 0}\frac{\delta u}{\ell}=\infty\quad h<1$$

Regularizing scale through viscosity:

$$\eta_h\sim\nu^{1/(1+h)}\sim Re^{-1/(1+h)}$$

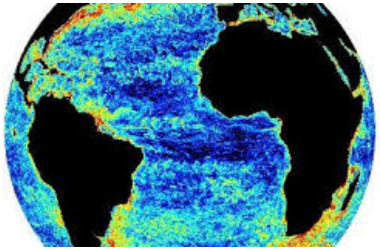
Paladin&Vulpiani

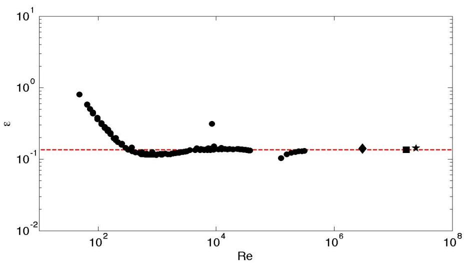


Singularity of NSE: h=-1
Quasi-singularity of NSE: h>-1

$$\lim_{\ell\to 0}\lim_{\nu\to 0}\delta u\sim\ell^h$$

Regularity for physicist: Dissipative vs non-dissipative singularity





"...in three dimensions a mechanism for complete dissipation of all kinetic energy, even without the aid of viscosity, is available."

L. Onsager, 1949

See Eyink&Sreenivasan (2006)

$$D(u)[x] \propto \lim_{\ell \to 0} \ell^{3h-1}$$

Inertial dissipation:

Local energy dissipation:

$$\epsilon_\ell = \frac{1}{4}\int d\vec{\xi}\, \nabla\phi^\ell(\xi)\cdot\delta_\xi\vec{u}(\delta_\xi\vec{u})^2 + \frac{\nu}{2}\int d\vec{\xi}\, \nabla^2\phi^\ell(\xi)(\delta_\xi\vec{u})^2.$$

$$\sim_{\ell\to 0} -D(\vec{u}) - \nu(\nabla\vec{u})^2$$

$$\frac{1}{2}\partial_t \mathbf{u}^2 + \mathrm{div}\left(\mathbf{u}\left(\frac{1}{2}\mathbf{u}^2 + p\right) - \nu\nabla\mathbf{u}\right) = D(u) - \nu\, \nabla\mathbf{u})^2$$

Duchon&Robert. Nonlinearity (2000), (

Link with Onsager's conjecture

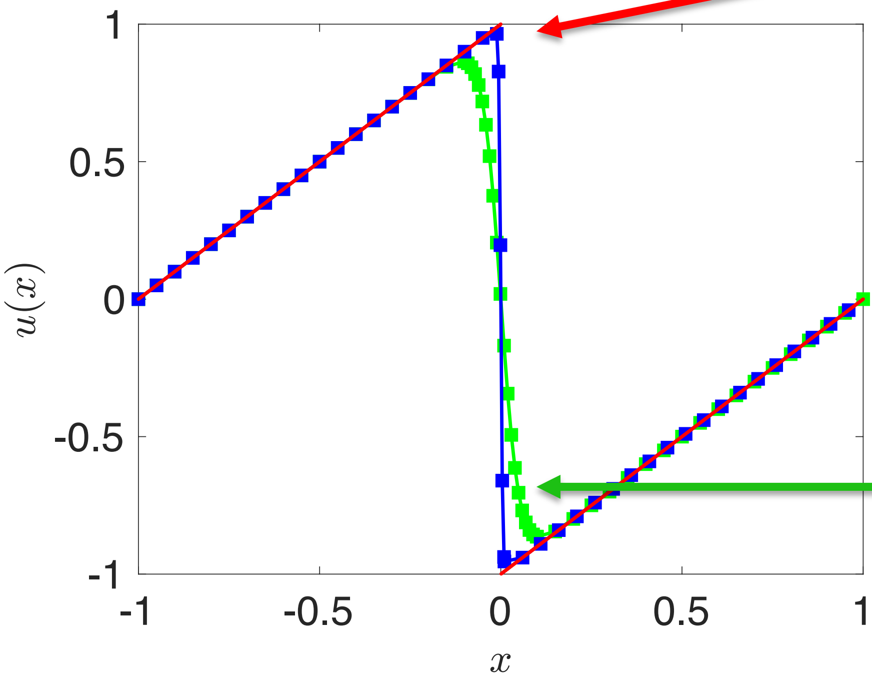
If $h > 1/3$ → Euler equation conserves energy,
Dissipation in Navier-Stokes by viscosity.
(Eyink 1994, Constantin et al, 1994)

If $h \leq 1/3$ → Dissipation through irregularities (singularities)
Without viscosity !

(Isett, 2018)

Singularity vs quasi-singularity: Illustration on Burgers solution

$$\partial_t u + u\partial_x u = \nu\, \partial_{xx} u$$



$O(\nu)$

$$\nu = 0$$

$$\delta u = \Delta u\ \ell^0$$

True singularity

$$h = 0$$

$$\nu > 0$$

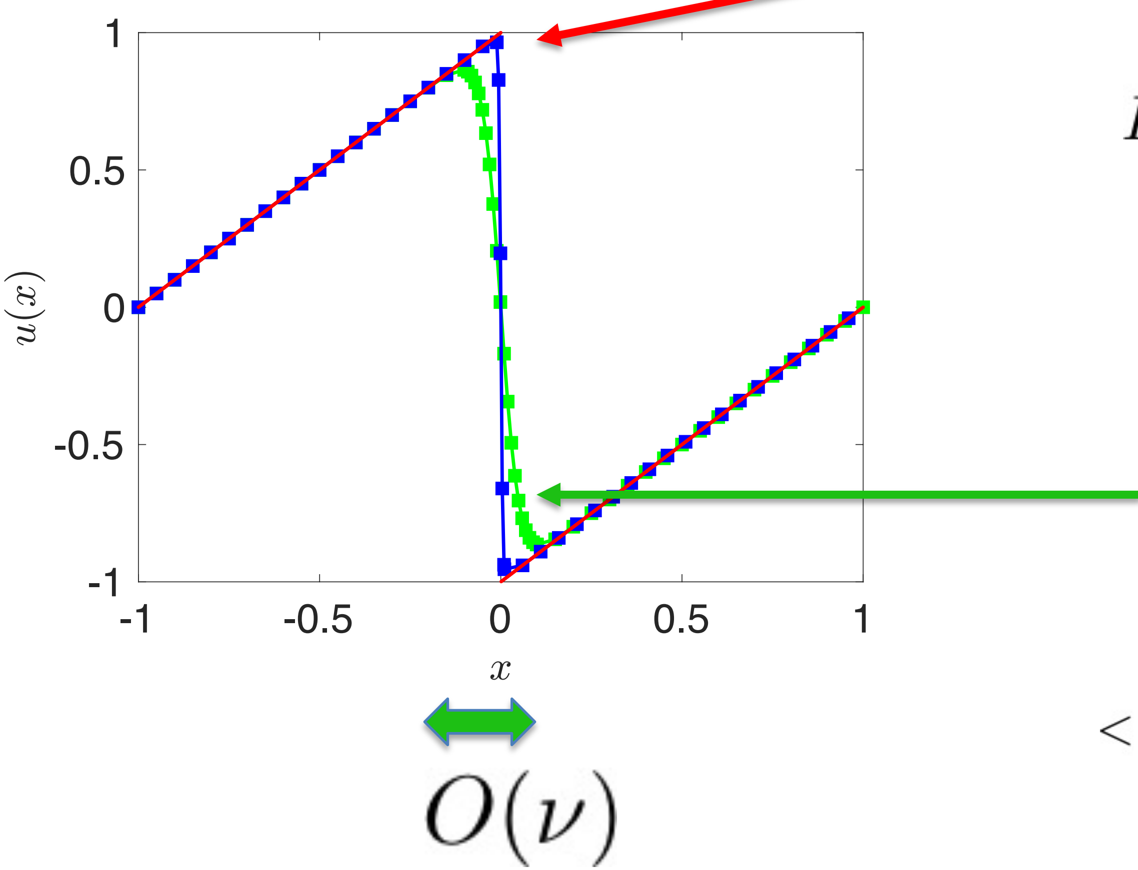
Quasi-singularity
Regularization over distance

$$\eta_0 \sim \nu$$

Dissipative singularity:
Illustration on Burgers solution

Eyink 2007-2008

$$\partial_t u + u\partial_x u = \nu\, \partial_{xx} u$$



$\nu = 0$

$D(u) = (\Delta u)^3/12L$

$\nu > 0$

$D(u) = 0$

$< \nu(\partial u)^2 >= (\Delta u)^3/12L$

Regularity for physicist:
Link with scaling symmetries?

Hölder continuity: $\exists(C,\epsilon,h),\ s.t.\ \forall\ell<\epsilon,\quad |\vec{u}\,(\mathbf{x}+\ell)-\vec{u}\,(\mathbf{x})\,|<C\ell^h.$

Building Blocks: velocity increments $\delta\vec{u}(\mathbf{x},\ell)=\vec{u}(\mathbf{x}+\ell)-\vec{u}(\mathbf{x})$

$$\delta\vec{u}=\vec{u}(x+\ell)-\vec{u}(x)\sim_{\ell\to 0}\ell^h$$

Link with scaling properties of NSE:

$$(t,x,u)\to(\lambda^2 t,\lambda x,\lambda^{-1}u)\quad\forall\lambda$$

Leray rescaling

Leray 1934

When $\nu=0$ **(Euler equation)**

$$(t,x,u)\to(\lambda^{1-h}t,\lambda x,\lambda^h u)\quad\forall(\lambda,h)$$

h-rescaling

Frisch, Book

Interesting questions: What are the values of h for NSE and Euler?

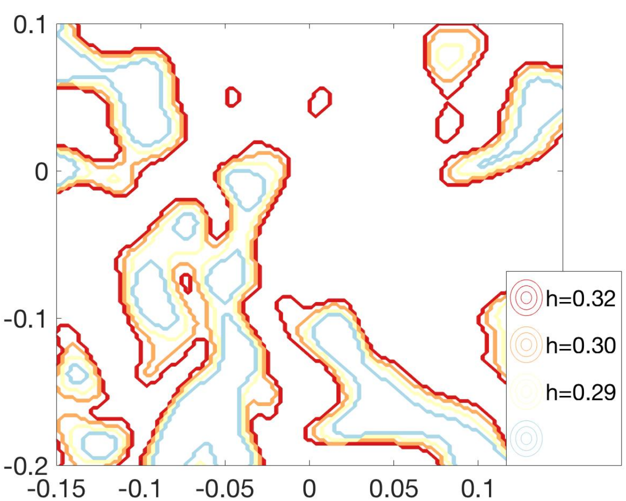
Probabilistic search using large deviations

Cafarelli theorem Singularities, if any, are very rare

Probabilistic search, using large deviations

$$\text{Prob}\left[\ln(\delta u) = h \ln(\ell/L)\right] \sim_{\ell \to 0} e^{\ln(\ell/L)C(h)} = \left(\frac{\ell}{L}\right)^{C(h)},$$

C(h): large deviation function of h

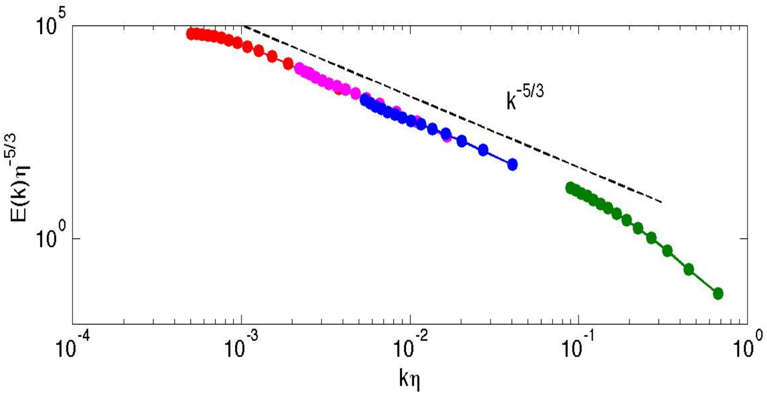


Heuristic interpretation of Parisi&Frisch (1987)

$\delta u \approx \ell^h$ over a fractal set of Codimension $C(h)$

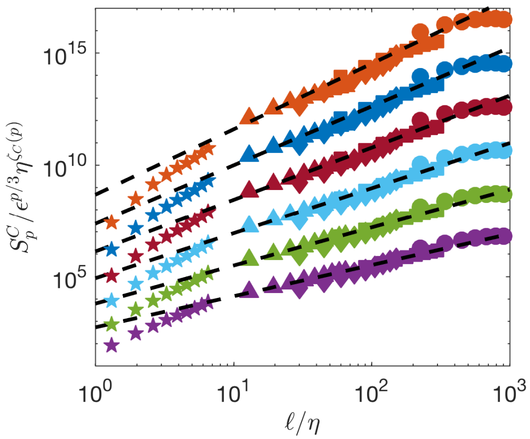
C(h) is also called multi-fractal spectrum

Expressing Physicists laws with C(h)



4/3 law

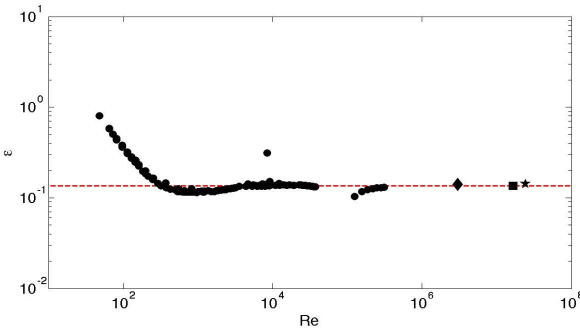
$$\langle (\delta u)^3 \rangle = \frac{4}{3}\epsilon\ell \sim \ell^{\zeta(3)}$$



Exponent Velocity Structure Functions:

$$\zeta(p) = \min_h (ph + C(h))$$

Parisi&Frisch, 1987



Mean energy dissipation:

$$\langle \epsilon \rangle \sim Re^{-\xi}$$

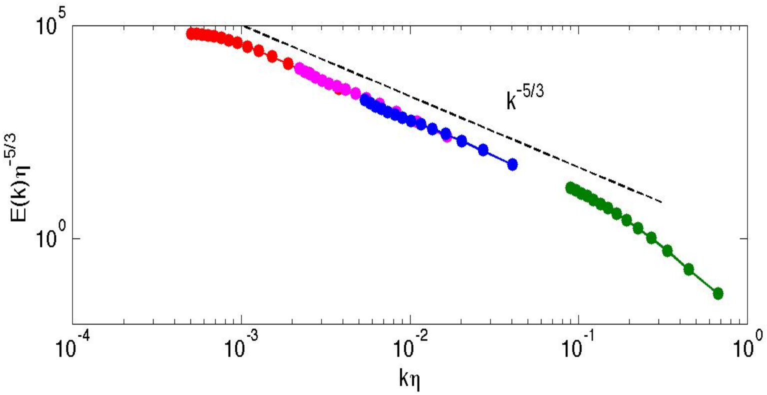
$$\xi = \min_h \left(\frac{3h - 1 + C(h)}{1 + h} \right)$$

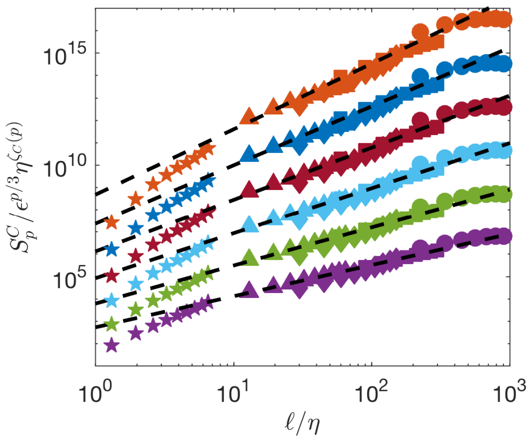
Boffetta et al, 2008

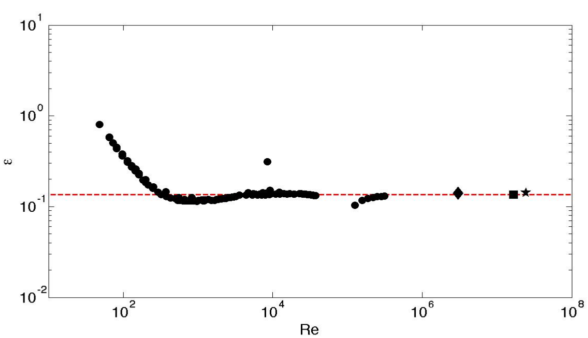
Nelkin, 1990

Benzi et al, 1991

Constraints on C(h)



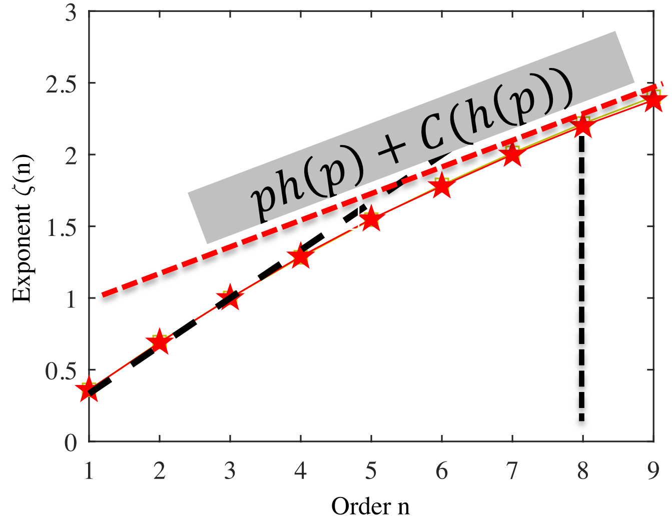




4/3 law

$$C(h) \geq 1 - 3h$$

Benzi &Biferale, 2009



Legendre Property

Parisi&Frisch, 1987

Mean energy dissipation:

Boffetta et al, 2008

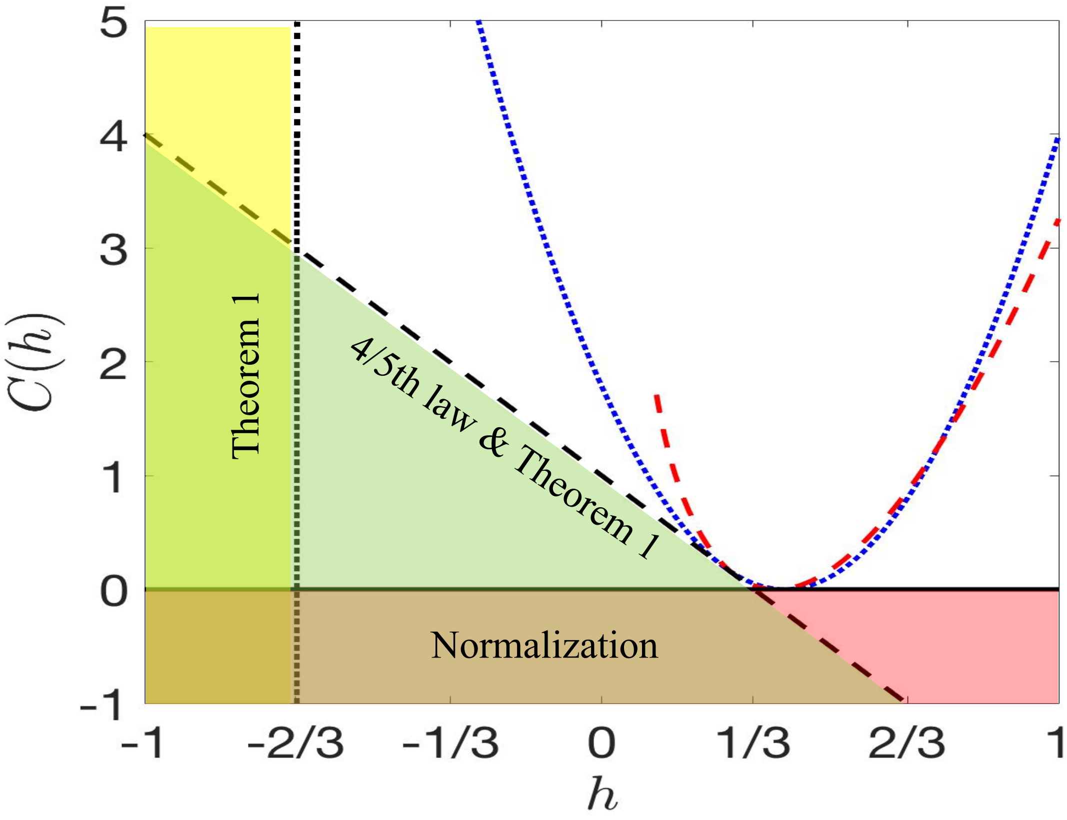
$$C(h) \geq 1 - 3h$$

Theoretical constraints using bounds on velocity gradient

Gibbon's theorem for weak solutions of NSE on a torus

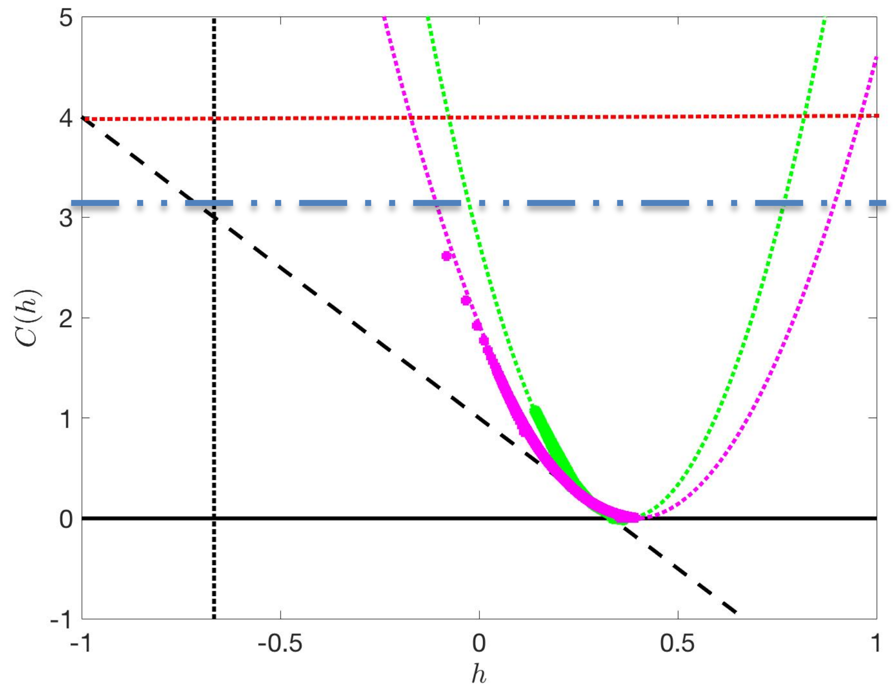
$$\left\langle \left(\nu^{-1} \|\nabla^n u\|_{2m}\right)^{1/\alpha_{n,m}} \right\rangle_T \leq c_{n,m}\, Re^3$$

$$\alpha_{n,m} = \frac{2m}{2m(n+1)-3}$$



Impossibility to reach h=-1 for periodic boundary conditions (***Dubrulle&Gibbon, 2021***)

Empirical constraints on C(h)



1 C(h) is independent of the flow forcing or geometry

Arneodo et al, 1996

2 C(h) is close to log-normal

$$C(h) = \frac{(h - 1/3 - 3b/2)^2}{2b}$$

Arneodo, Chevillard, Castaing…

3 C(h) depends on the velocity components

Kestener&Arneodo, 2004

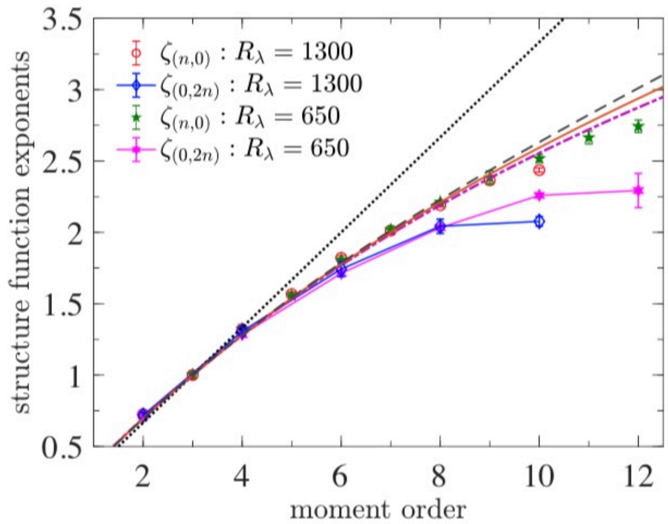
b=0.025 for longitudinal velocity increments

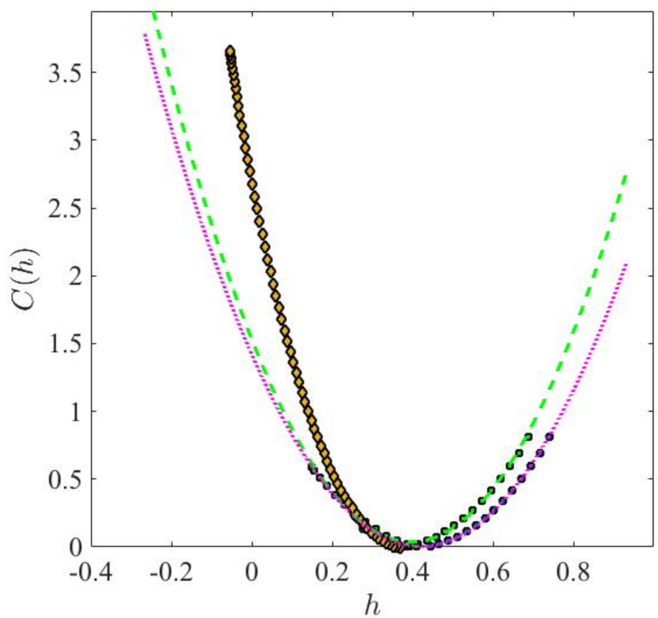
$$\delta u_T = \vec{\ell} \cdot \vec{u}(x+\ell) - \vec{u}(x))/\ell^2$$

b=0.04 for transverse velocity increments

$$\delta u_T = \vec{\ell} \times \vec{u}(x+\ell) - \vec{u}(x))/\ell^2$$

Empirical constraints on h_{min}





Periodic bc

$h_{min} > 0$ for longitudinal velocity increments

$h_{min} = 0$ for transverse velocity increments

Iyer, Sreenivasan&Yeung, 2020

Non periodic bc

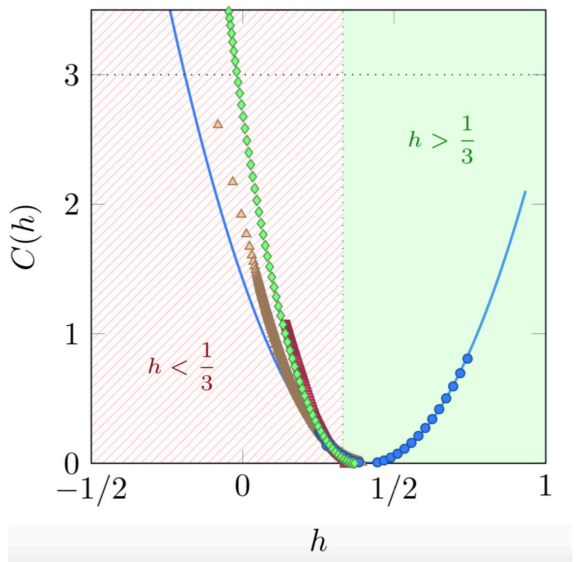
Use refined similarity to extend range

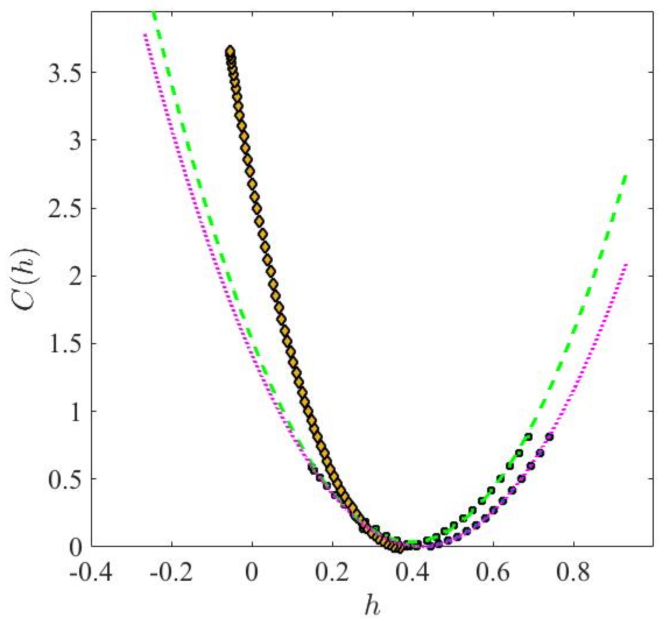
$$\frac{\langle|\delta u_\ell|^p\rangle}{\langle|\delta u_\ell|^3\rangle^{p/3}} = \frac{\langle|D_I^\ell|^{p/3}\rangle}{\langle|D_I^\ell|\rangle^{p/3}}$$

$$h_{min} <\sim 0$$

Faller et al, 2021

Empirical constraints on h_{min}





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$$h_{min} <\sim 0$$

Faller et al, 2021

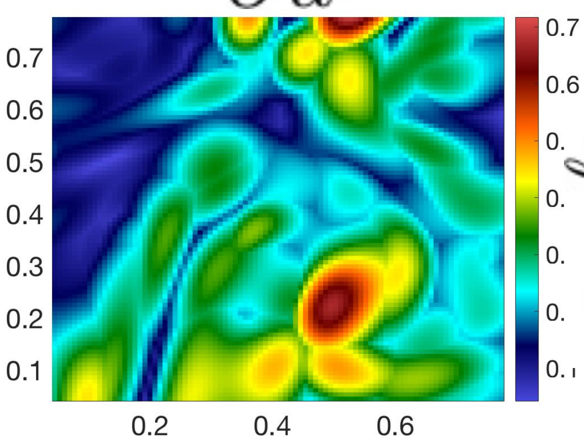
Deterministic search using multifractal

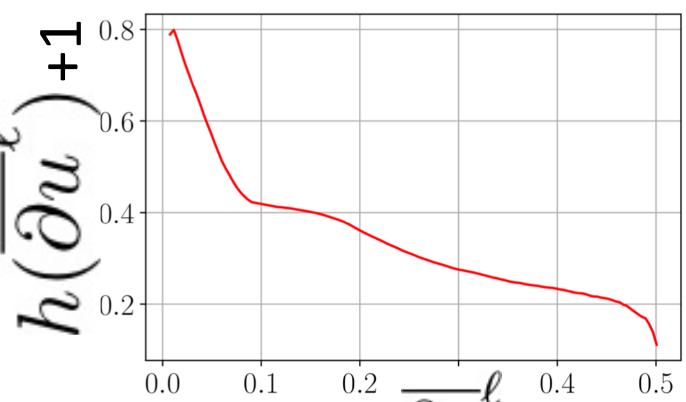
$$\overline{\partial u}^{\ell} \sim \frac{\delta u}{\ell} \sim \ell^{h-1}$$

C(h) provides a statistical estimate on the strength of the filtered large gradients and can be used to map locally the Holder exponent

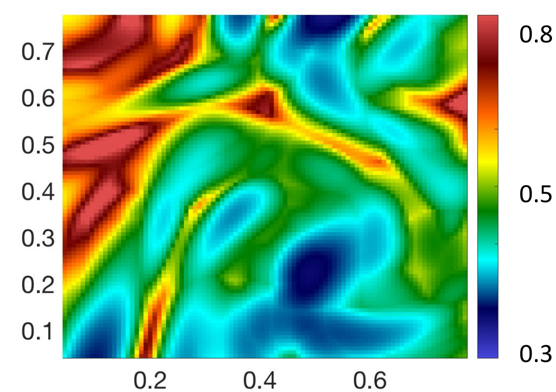
$$h(p)\log(\ell) \sim \frac{\langle log(\delta u)\,(\delta u)^p\rangle}{\langle(\delta u)^p\rangle}$$

$$\overline{\partial u}^{\ell}$$





Pseudo-Holder

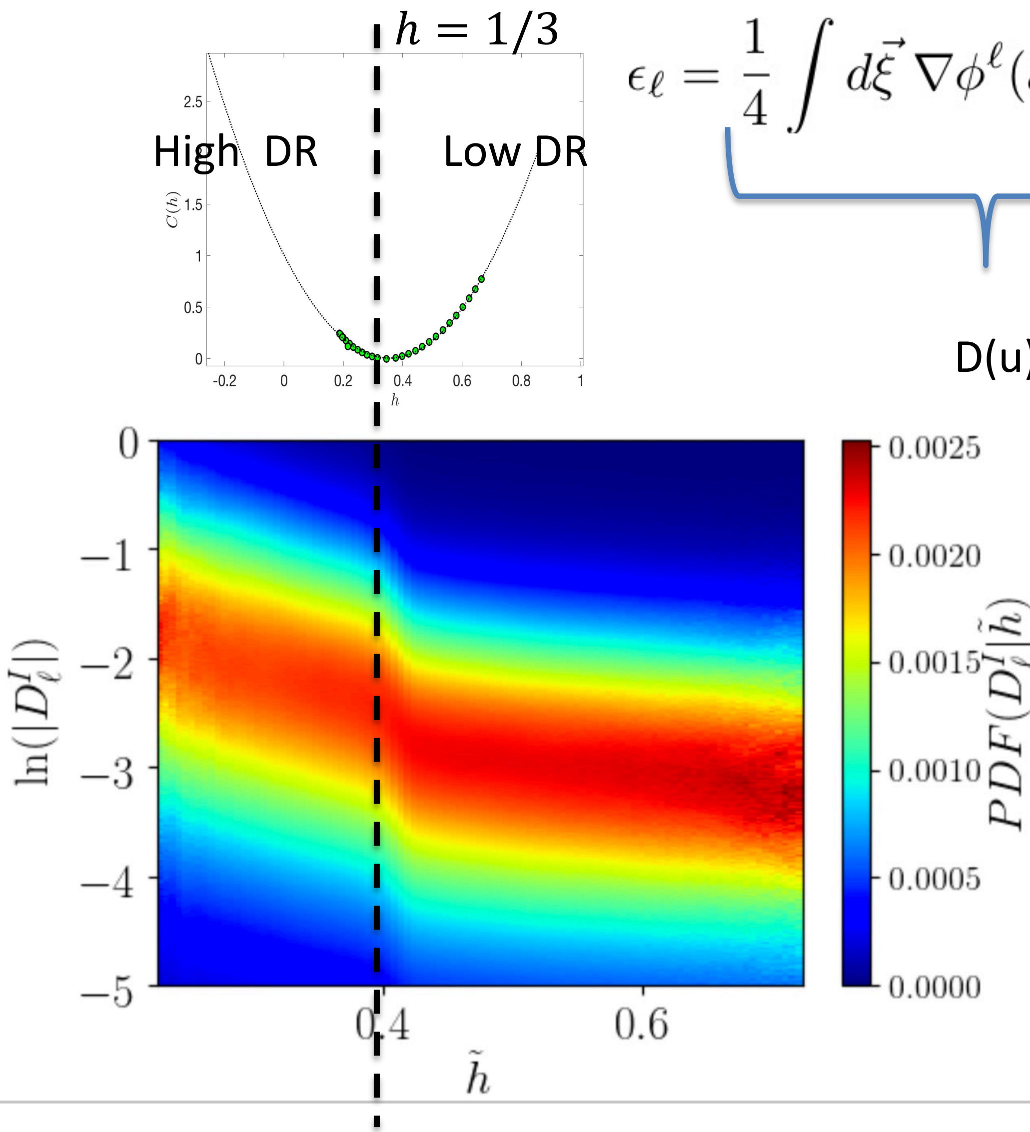


Nguyen et al, PRE (2019)

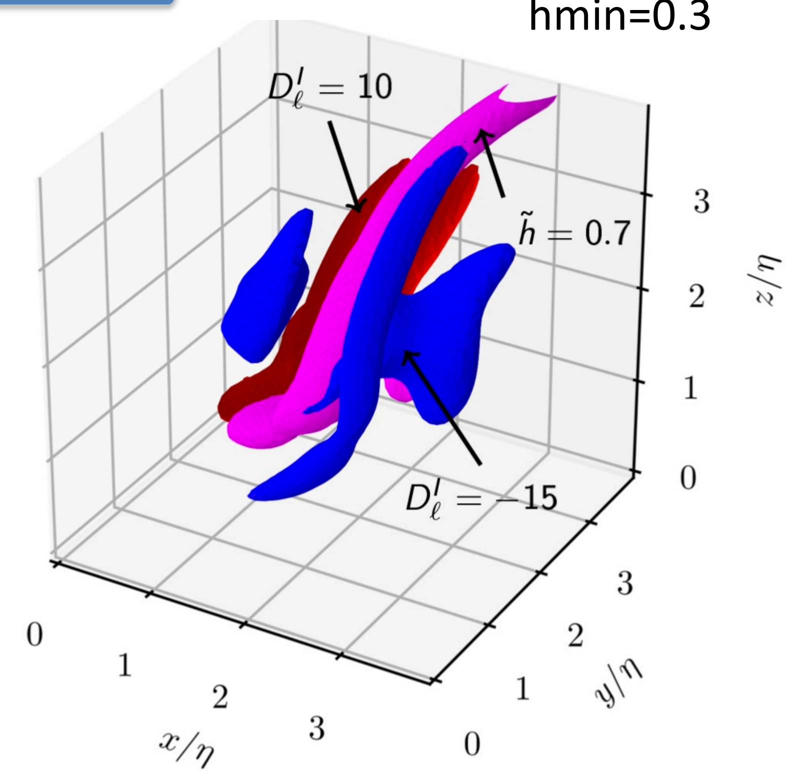
Pseudo-Holder vs D(u)

$$\epsilon_\ell = \frac{1}{4}\int d\vec{\xi}\,\nabla\phi^\ell(\xi)\cdot\delta_\xi\vec{u}(\delta_\xi\vec{u})^2 + \frac{\nu}{2}\int d\vec{\xi}\,\nabla^2\phi^\ell(\xi)(\delta_\xi\vec{u})^2$$

D(u)



Statistical correlation

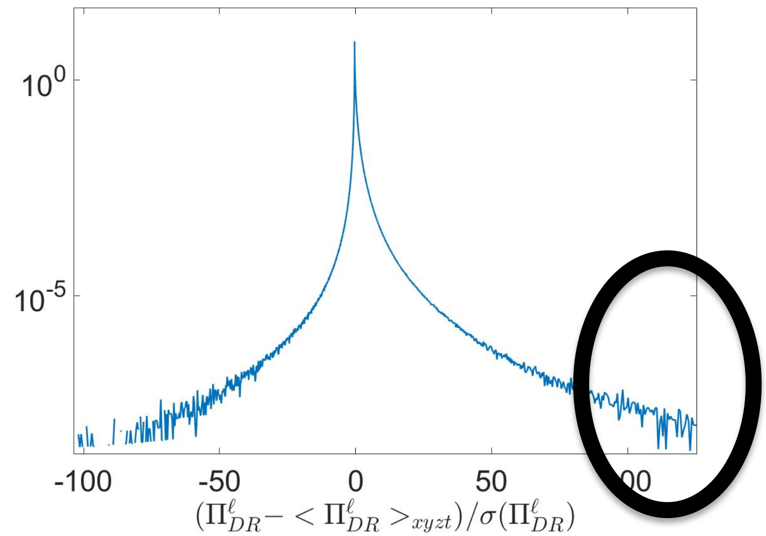


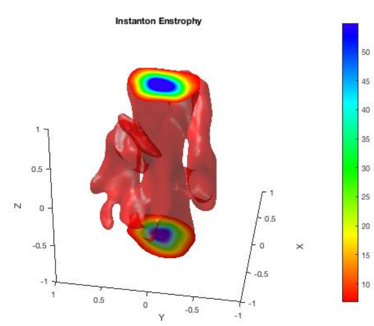
Temporal and spatial correlation

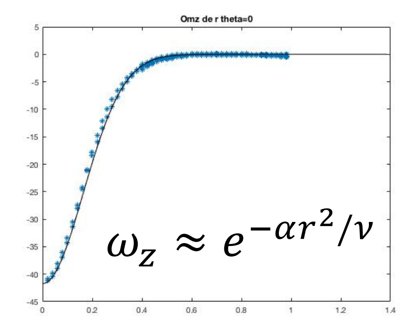
Nguyen et al, PRE (2019)

Statistics and geometry of most irregular events

$$D_\ell(\mathbf{u}) = \frac{1}{4}\int_{\mathcal{V}} d^3r\ (\boldsymbol{\nabla} G_\ell)(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r})\ |\delta\mathbf{u}(\mathbf{r})|^2,$$







The strongest events are Burgers vortices

Nguyen PhD Thesis (2020); ; Nguyen et al, PRE (2020)

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