École des Ponts
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Multifractal analysis and simulations: from WAUDIT to RW-Turb

by Daniel Schertzer

Hydrology, Meteorology & Complexity (HM&Co)

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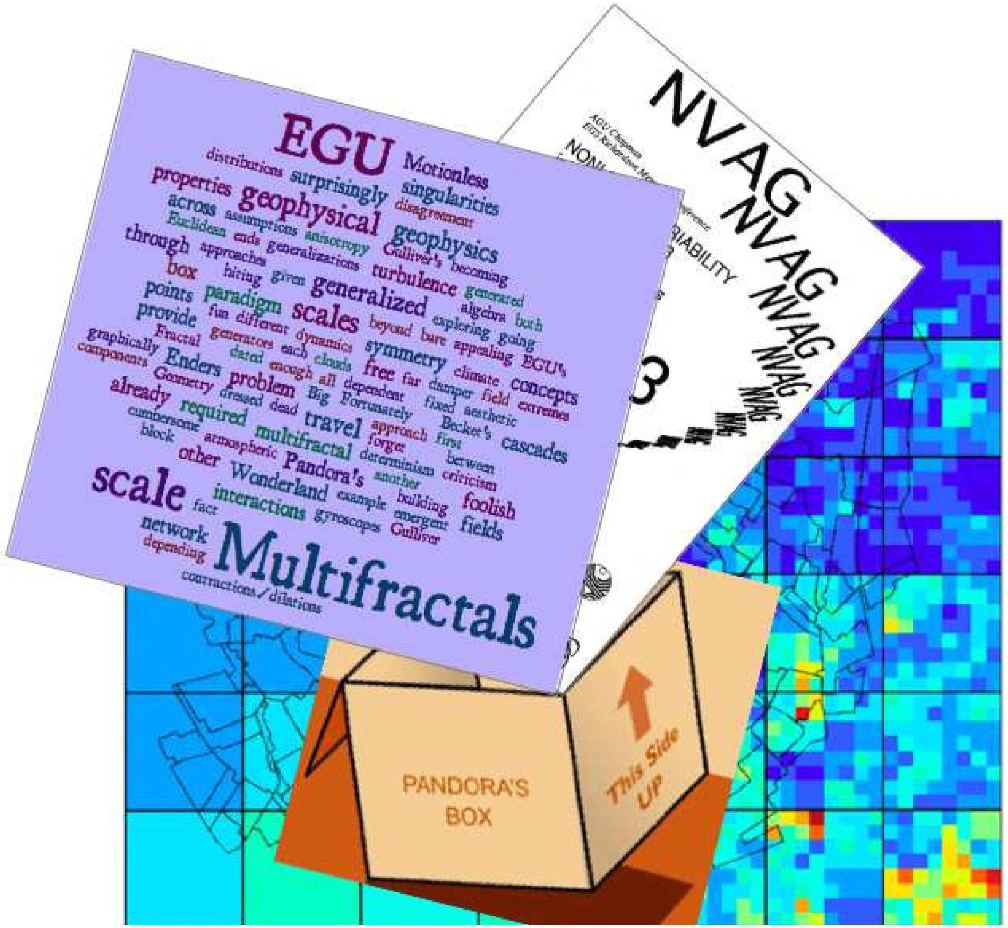

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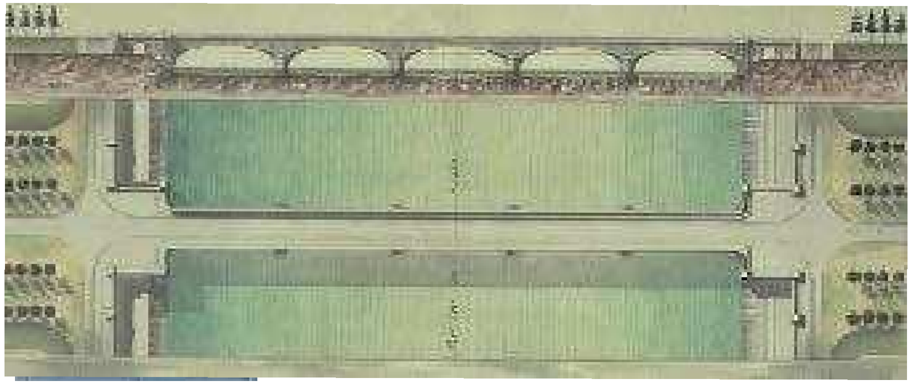

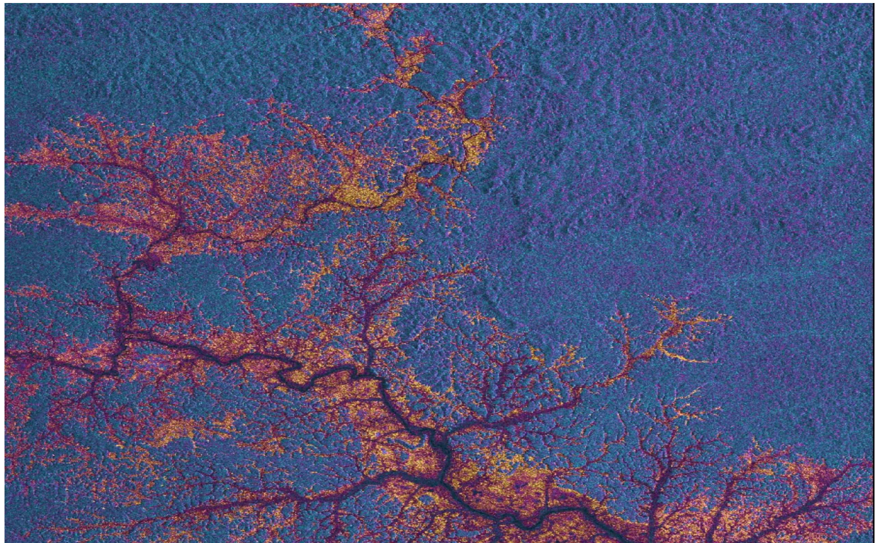

ERS view of the Amazon basin

- Formerly Ecole Nationale des Ponts et Chaussées ("National School of Bridges and Roads")
  - one of the world's oldest Civil Engineering School (1747)
  - Cauchy, Carnot, Coriolis, Darcy, Fresnel, Navier, Saint-Venant… Liouville
    - Becquerel (Nobel in Physics), J. Tirole (Nobel in Economy)
- last decades
  - beyond its more traditional fields and into an **international institution**,
  - adapting to the changing demands of the **modern world**
    - cofounder of ParisTech Paris-Est University and AvanCity
    - **teaching complex systems, multifractals,** etc. to young generations of engineers and managers

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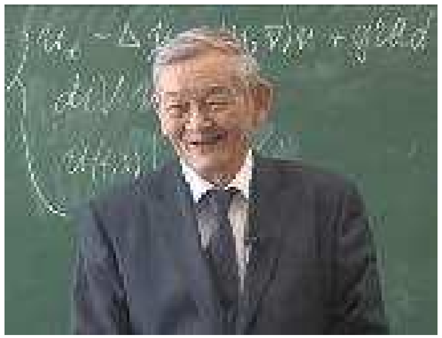
From Navier (1822)… to Stokes (1843)

Louis Navier

Augustin-Louis Cauchy

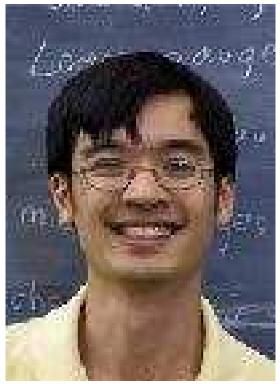
Adhémar Jean Claude Barré de Saint-Venant

Sir George Stokes



A **millenium problem** raised at Ecole Nationale des Ponts et Chaussées, recent episodes:

Otelbaev (2013) and Tao (2015)



Difficulties of the subject

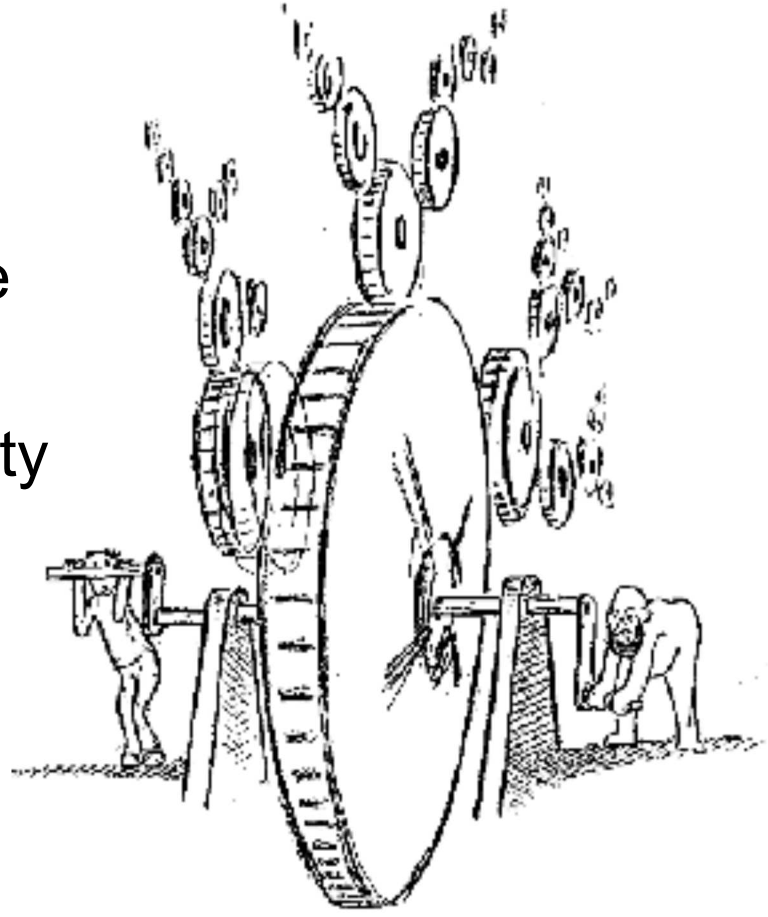
'Turbulence'

Maths: Existence? Uniqueness?
Navier-Stokes solutions for D > 2:

- ➢ we are still at the level of conjecture (Leray 1933, 1934)
- ➢ major difficulty: quadratic nonlinearity

Physics: basically two laws, relatively equivalent, are the source of most developments:

- ➢ diffusion law of Richardson (1926),
- ➢ Kolmogorov spectrum (1941)



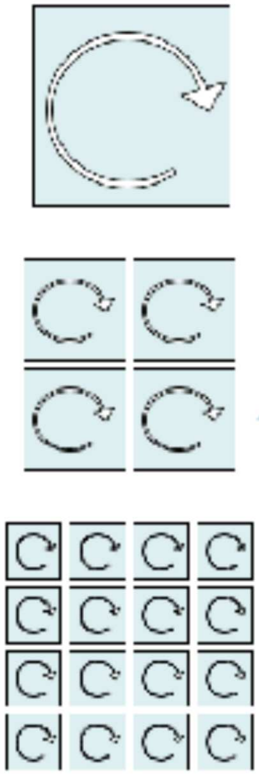
Ещё один
механизм турбулентности

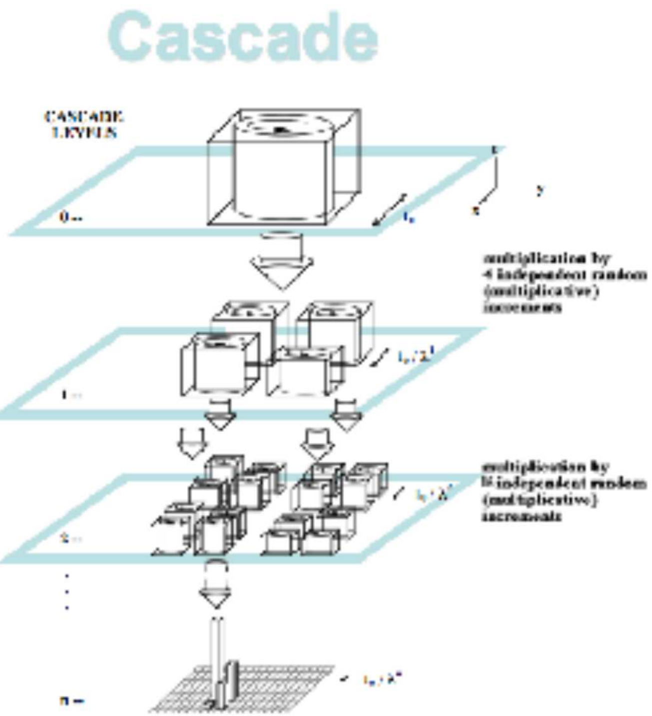
Rain & turbulence

Turbulences

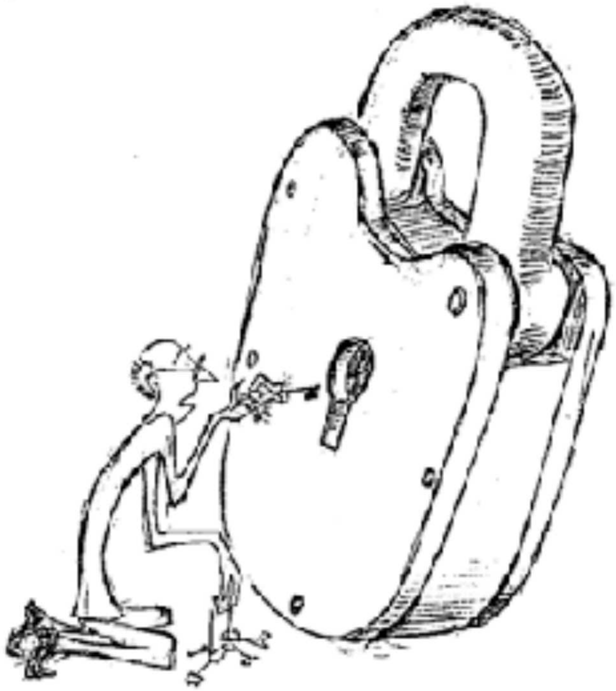
=> Modelling of rainfall intensity by multiplicative cascades

Solar energy	Water vapor	Clouds	Rain
	Localised energy	Localised energy +	Localised energy + +



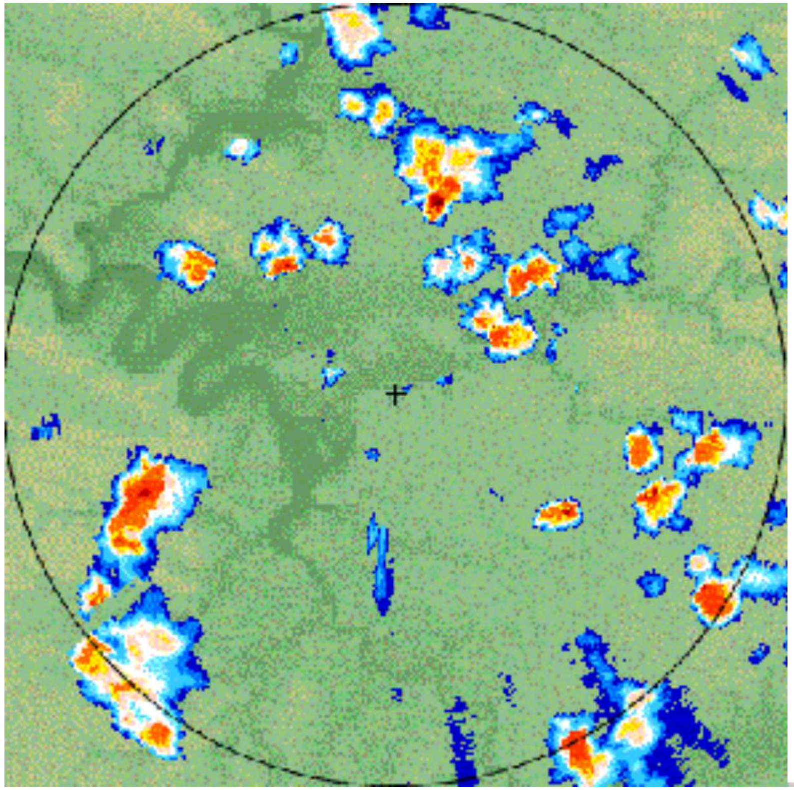


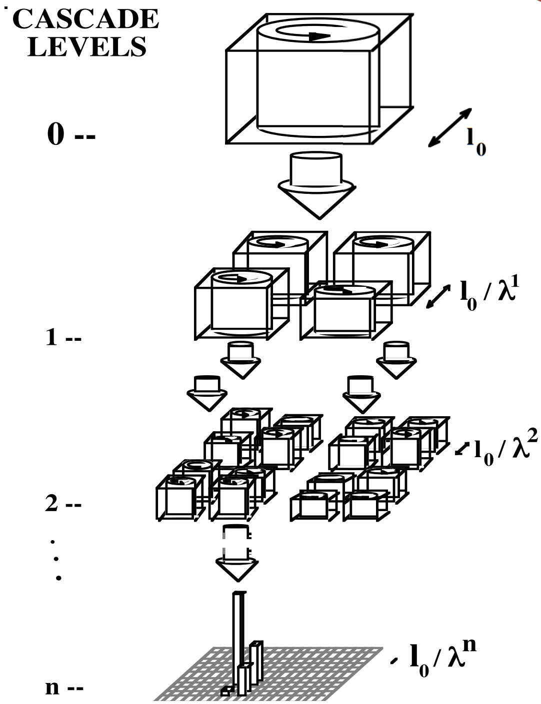
w.r.t. the scale ratio $\lambda = L / \ell$



Russian dolls… and multiplicative cascades

Discrete in scale cascades only for pedagogy !





multiplication by 4 independent random (multiplicative) increments

multiplication by 16 independent random (multiplicative) increments

Polarimetric radar observations of heavy rainfalls over Paris region during 2016 spring (250 m resolution):

- **heaviest rain cells** are much smaller than **moderate ones**
- true for their dimensions => **multifractal field**
- **complex dynamics** of their aggregation into a large front

Van der Hoven wind (integrated) spectrum (1957)

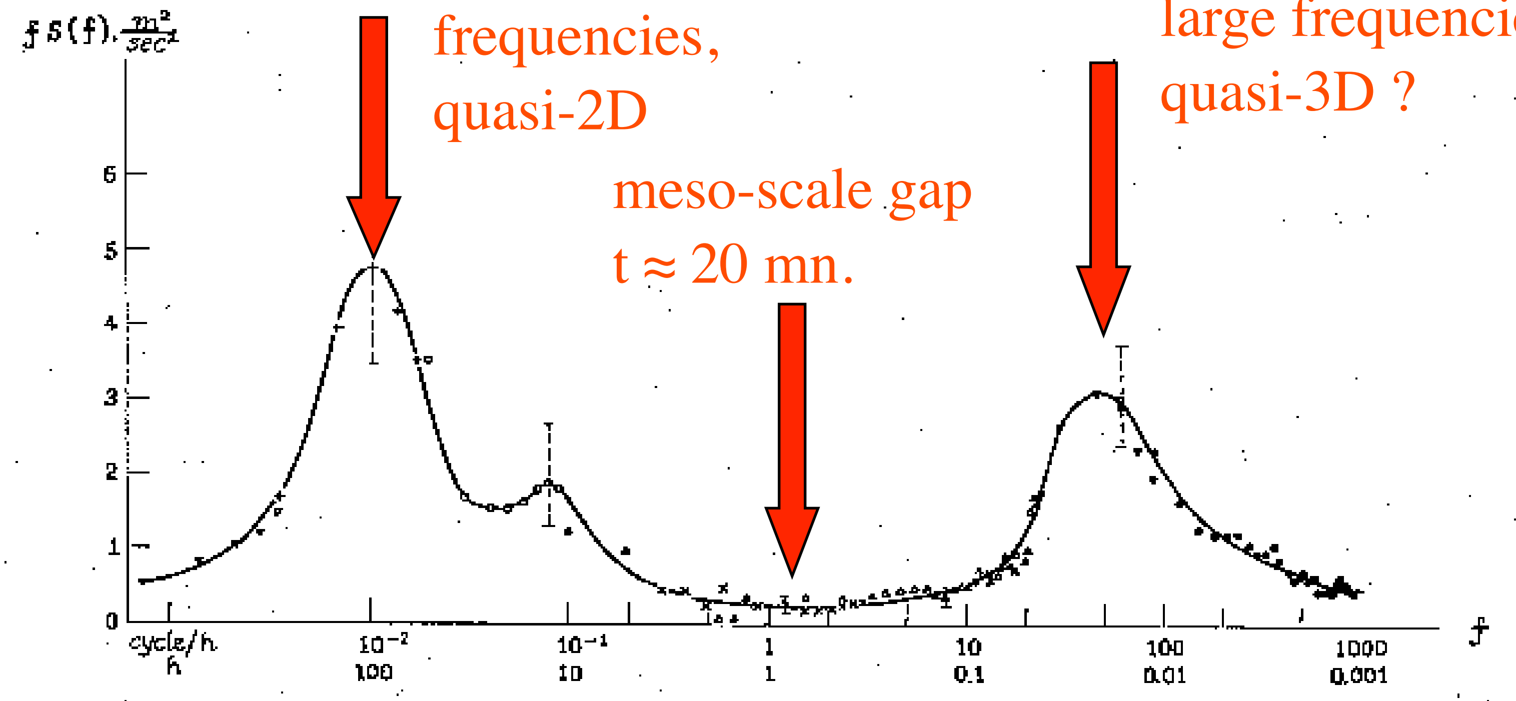


Fig. 3 Spectrum of the horizontal wind velocity. After Van der Hoven.[26] Some experimental points are shown on the graph; see reference 26.

various measurement devices

Richardson cascade is split into macro, meso, micro oscillations...

Physical scales vs. Euclidean scales

(e.g. 23/9-D atmospheric turbulence)

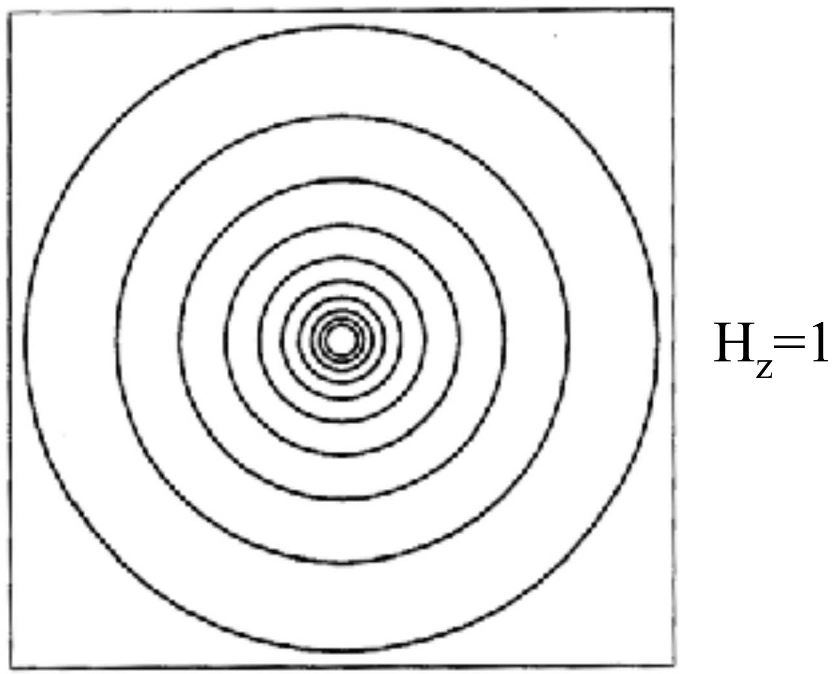
- stratification
  - flattening of structures at larger scales
  - “pancake turbulence”

$$\|x\|_p = \left(\sum_i |x_i|^{p/H_i}\right)^{1/p}$$

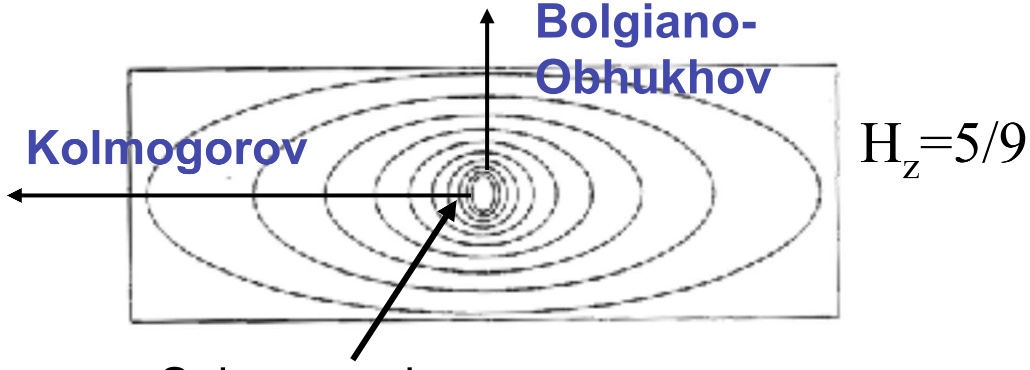
exemple:

$$H = (1, 1, H_z);\ p = 2$$

Isotropic Euclidean scales

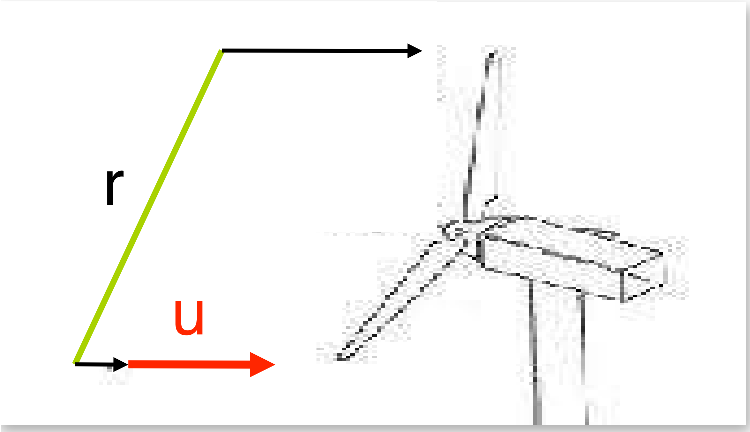


Anisotropic physical scales

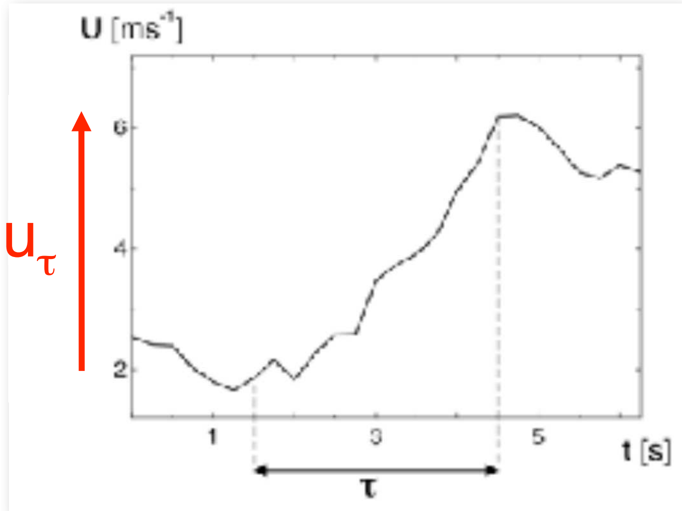


EU-INT WAUDIT research project

Wind Turbulence: Scales, Intermittency & Extreme events



$$u_r = U(x+r) - U(x)$$



$$u_\tau = U(t+\tau) - U(t)$$

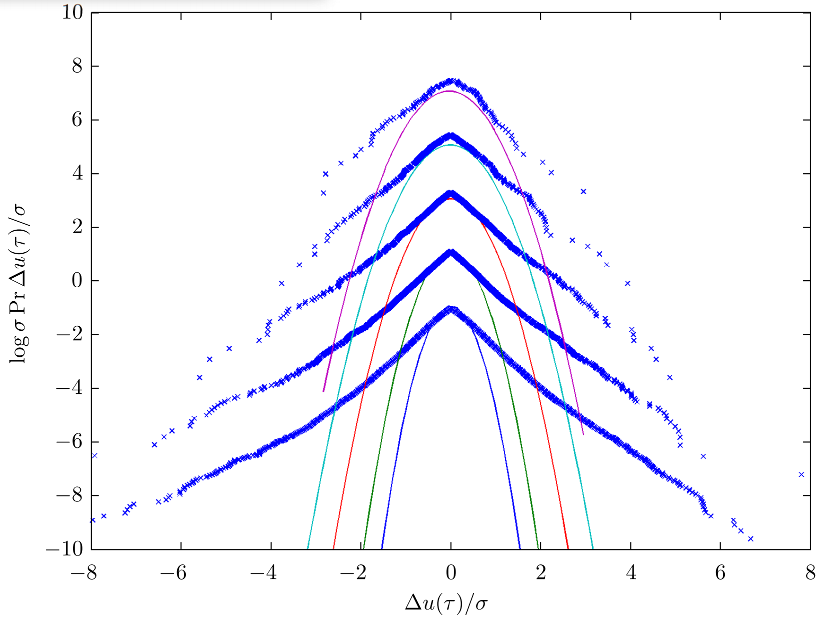


Figure 1.2: Same distribution as in figure 1.1 but with a logarithmic vertical axis and for increasing time-scales: $\tau = 0.1, 0.4, 1.6, 6.4$ seconds.

Experimental Data

Growian experiment (Germany):

Flat, coastal terrain
Two 150m masts

Data:
- Wind speed / direction (propeller)
  @ 10, 50, 75, 100, 125 and 150m
- Temperature:
  @ 10, 50, 100 and 150m

2.5Hz / 20 min
300 measuring runs
131 validated runs

Experimental Data

Corsica experiment:

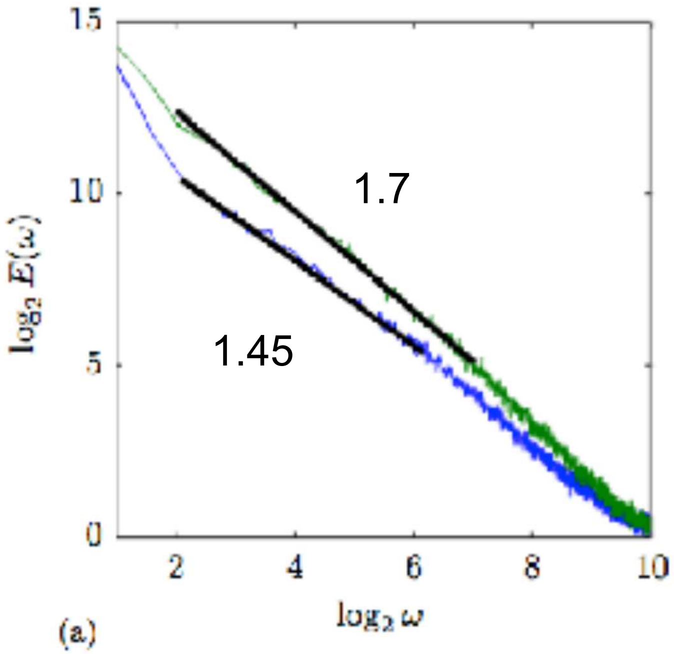
Mountainous, coastal terrain
One 43m mast

Data:
- Wind speed (sonic)
  @ 22, 23 and 43m
- Temperature:
  @ 22, 23 and 43m

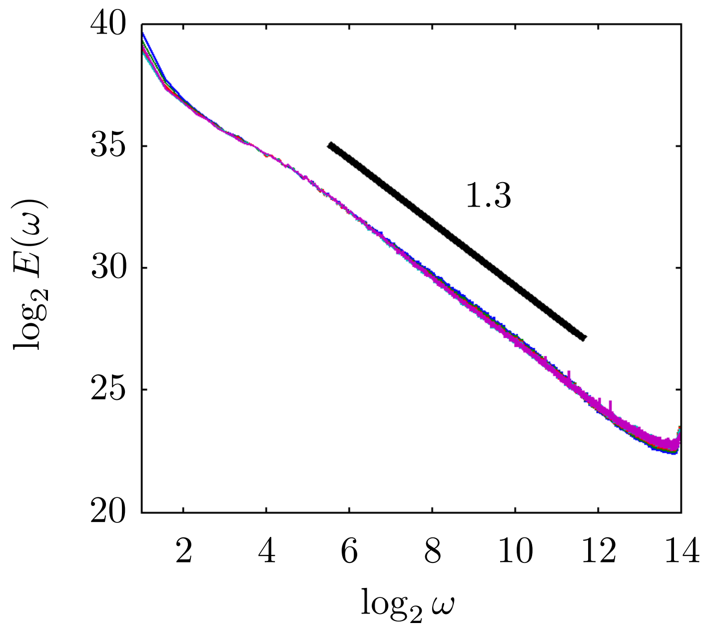
10Hz / 16h
180 measuring runs
161 validated runs

Spectral Analysis

Growian data



Corsica data

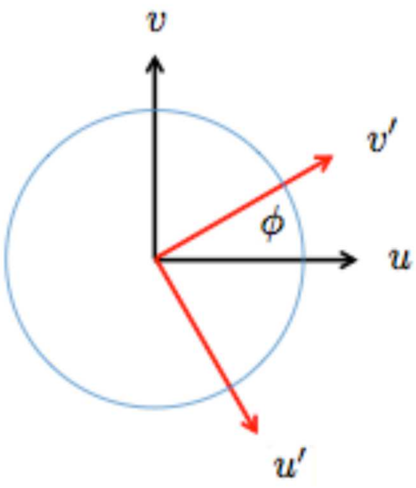


➢The combination of processes results in a scaling, statistical anisotropy.

Rotated Vectors' Energy Spectra

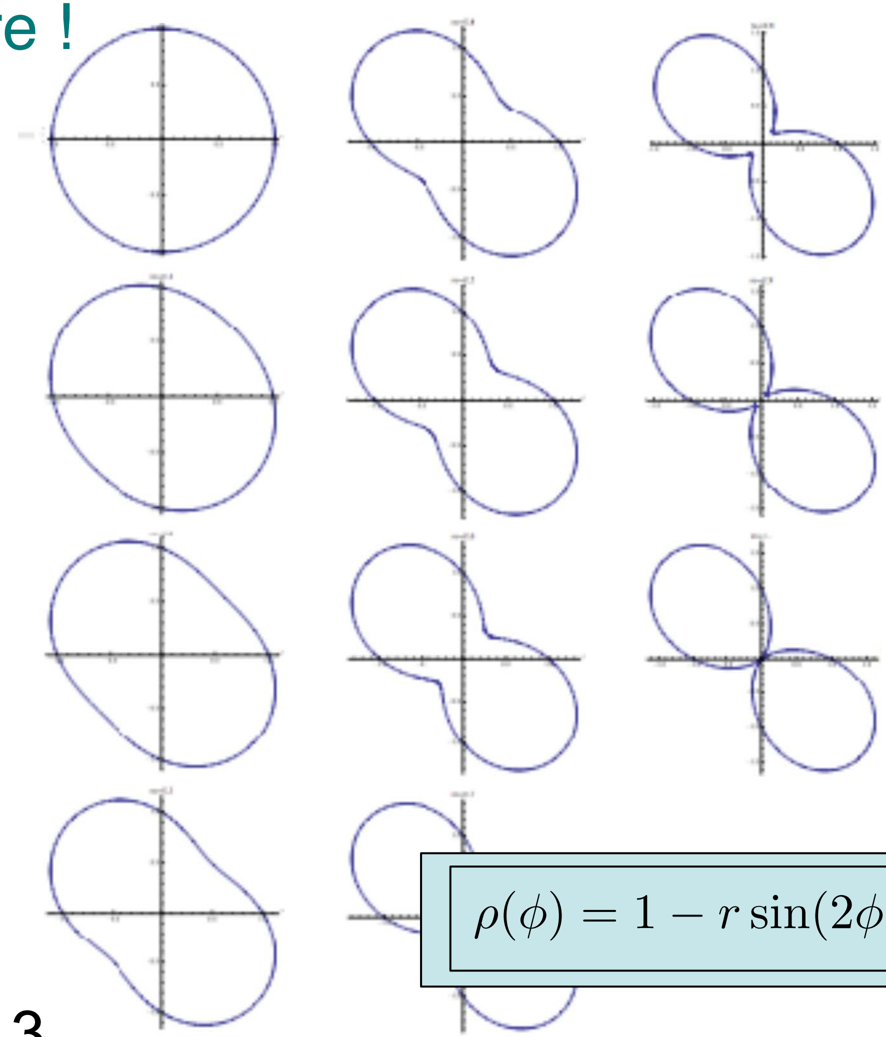
$$|\widehat{u'_\phi}(\omega)|^2 = E_\phi(\omega) = \cos^2(\phi)E_0(\omega) + \sin^2(\phi)E_{\pi/2}(\omega) - \sin(2\phi)E_{u,v}(\omega);$$

Inverse of the usual procedure !



Rotated time-dependent
u-component (Fourier space):

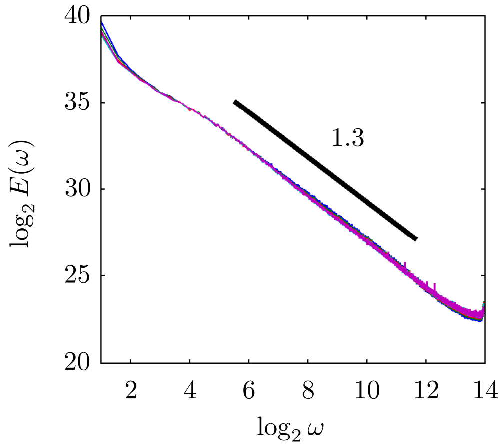
$$\widehat{u'_\phi}(\omega) = \cos(\phi)\hat{u}(\omega) - \sin(\phi)\hat{v}(\omega).$$

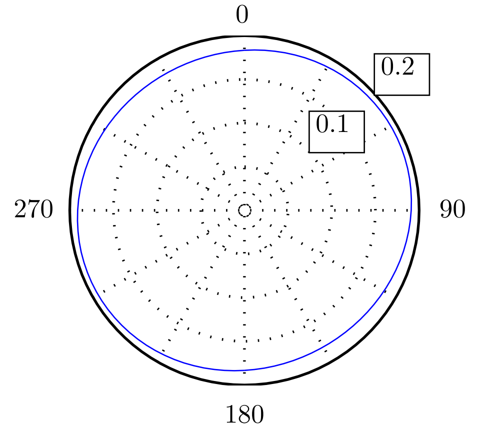


$$\rho(\phi) = 1 - r\sin(2\phi),$$

Scaling Anisotropy (Corsica)

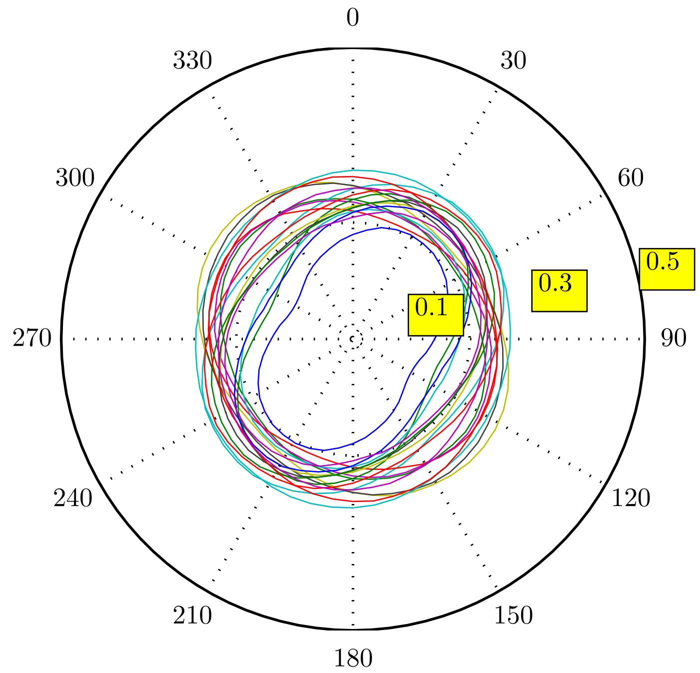
Scaling appears isotropic (with $H < 1/3$) on ensemble averaged spectra





Component-wise anisotropic for individual samples

Ensemble averaging the spectra results in a more isotropic rho function



- When the temperature is an active scalar i.e. $\Delta H > 0$, over larger scales (> 10 seconds) wind scaling anisotropy entails

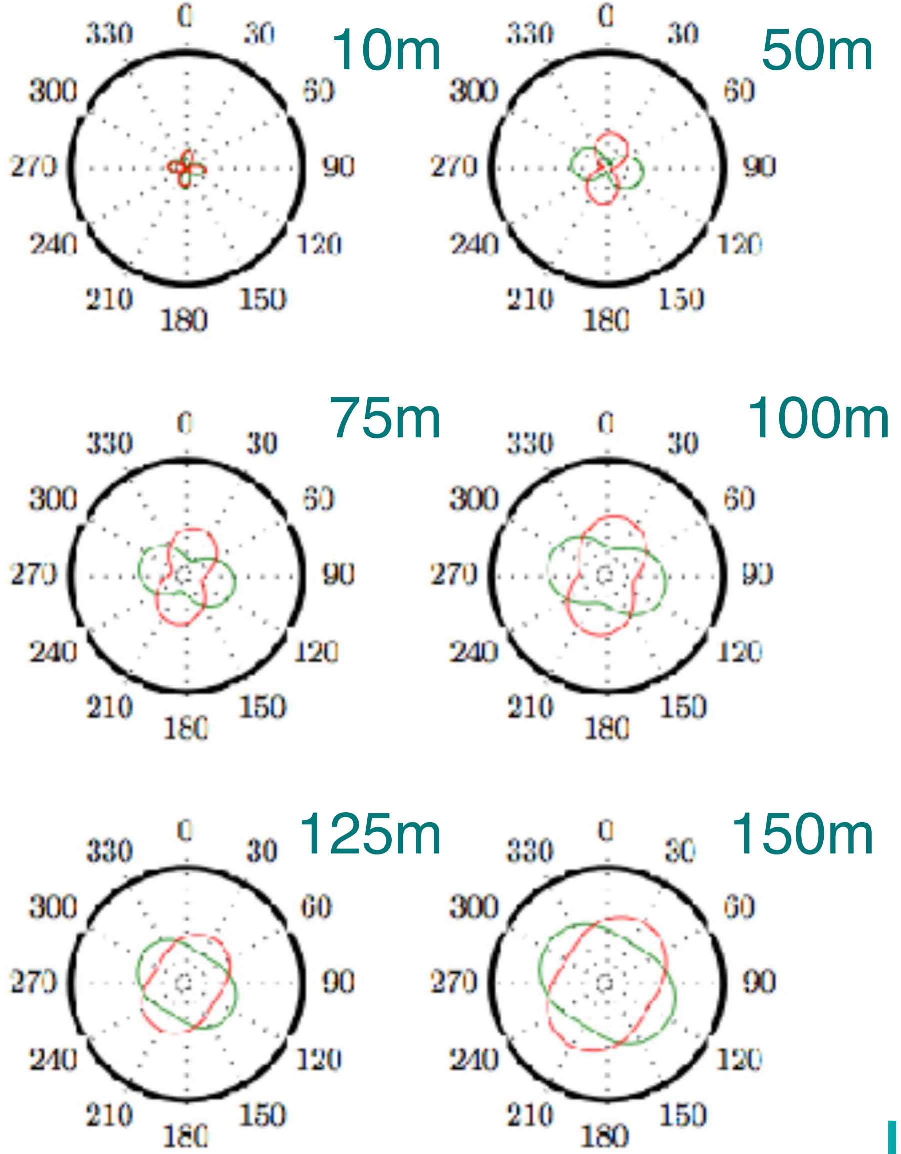
Scaling Anisotropy (Growian)

Isotropic homogeneous
$H = 1/3$

Empirical anisotropy
H_u and H_v

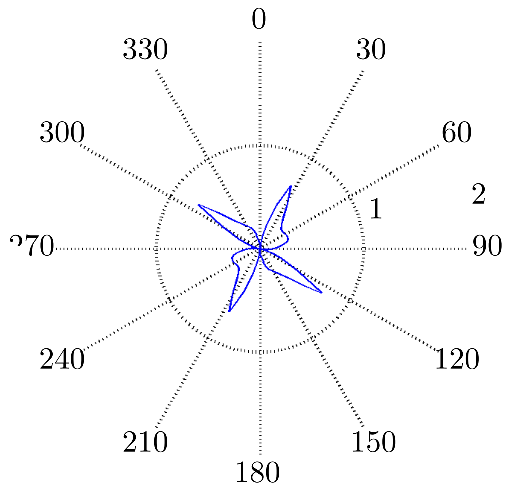
Rotation with height of the point of statistical isotropy ($H_u = H_v$)

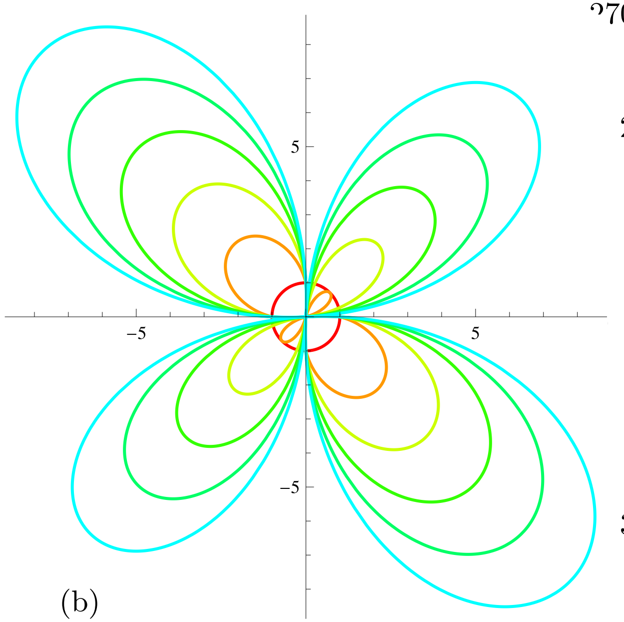
➢Scaling behaviour appears to be more complex over flat terrain

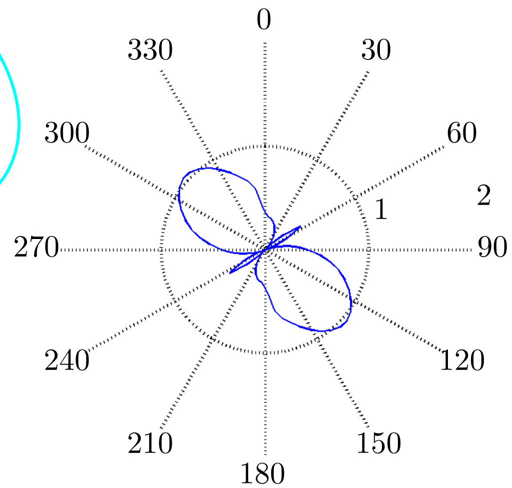


Scaling Anisotropy In The Wake

Analyses in the wake of the wind turbine yield four leaved structures (flowers) on individual fields i.e. highly anisotropic







These flowers can be constructed by using a correlation coefficient, r > 1.

Scale symmetries

Cascade paradigm

- ➢does not need to assume isotropy
- ➢but may lead to other non-trivial symmetries

The theoretical aspects were raised or supported by many empirical studies and have many consequences:

➢our overall understanding of Turbulence & Geophysics

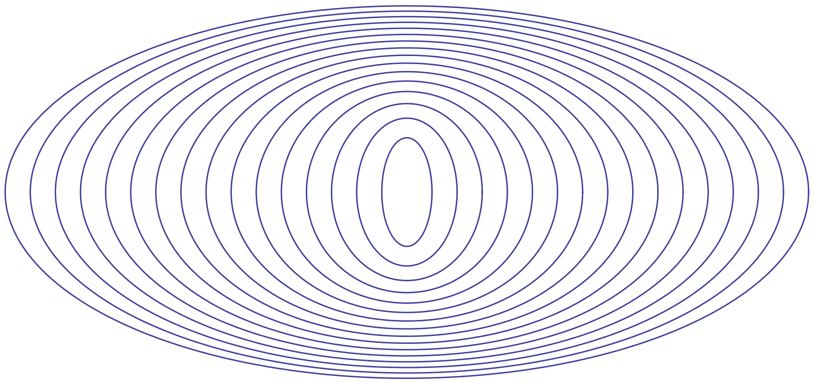
➢vast projects:

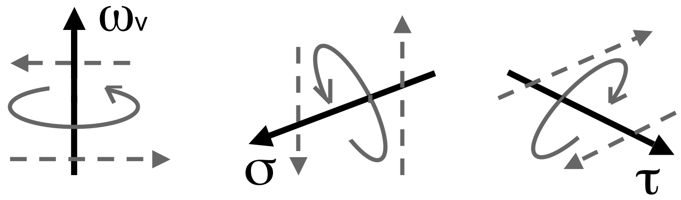
- ✓ space-time processes, vector fields
- ✓ deterministic / stochastic
- ✓ data treatment

$2+H_z$-dimensional vorticity equation

($0<H_z<1$)

Stratified atmosphere:





$$D\vec{\sigma}/Dt = (\vec{\sigma}\cdot\vec{\nabla}_h)\vec{u}_h$$

$$D\vec{\tau}/Dt = (\vec{\tau}\cdot\vec{\nabla}_h + \boxed{\vec{\omega}_v\cdot\vec{\nabla}_v)\vec{u}_h}$$

$$D\vec{\omega}_v/Dt = (\vec{\tau}\cdot\vec{\nabla}_h + \vec{\omega}_v\cdot\vec{\nabla}_v)\vec{u}_v$$

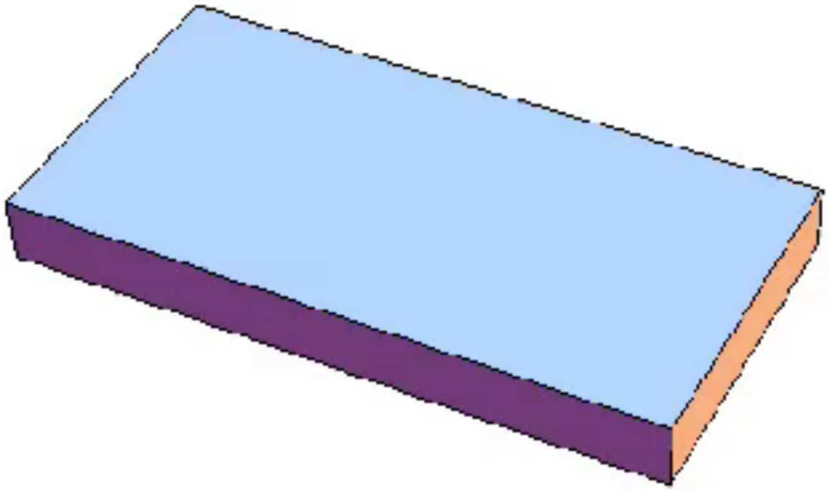
Strong interactions between *local generalized* scales,

= *strongly non local* (Euclidean) scales !

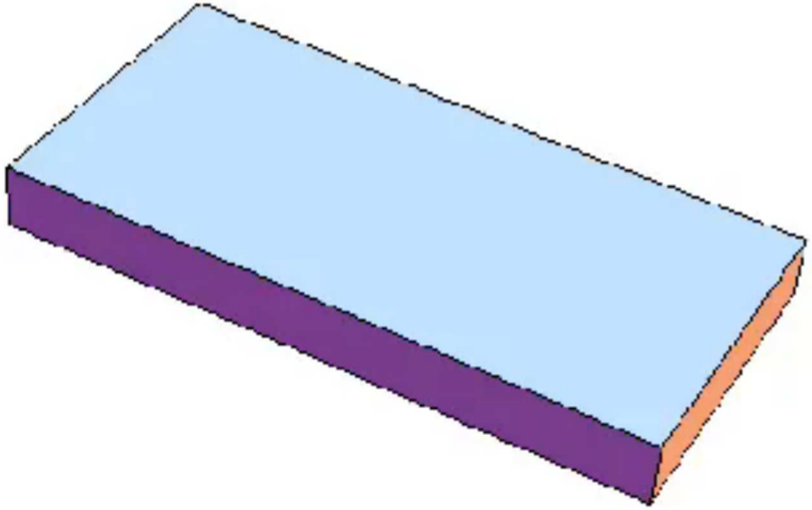
- a difficulty for direct numerical simulations ?
- easy for stochastic simulations !

wobble & gimbal lock

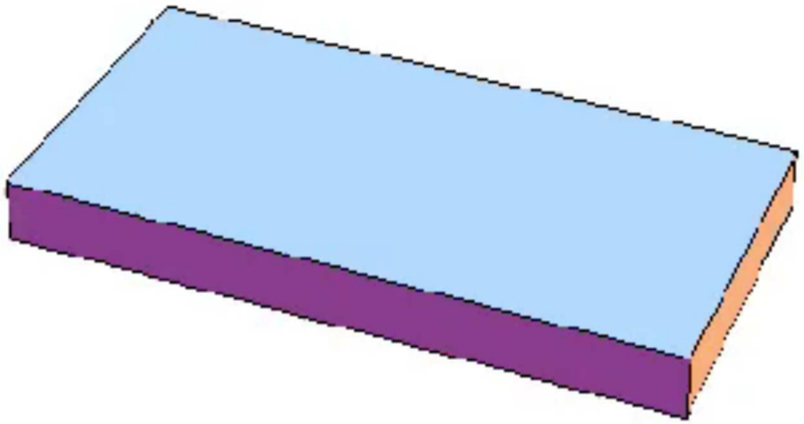
initial rotation along the main axis



initial rotation along the medium axis



initial rotation along the smallest axis



Hyperion

Apollo 13 guidance computer console

3D Scaling Gyroscope Cascade

$$\left(\frac{d}{dt}+\nu k_n{}^2\right)\hat{u}_n^{i}{}^*=i\{k_{n+1}[\left|\hat{u}_{n+1}^{2i-1}\right|^2-\left|\hat{u}_{n+1}^{2i}\right|^2]+(-1)^i k_n\hat{u}_n^{i}{}^*\hat{u}_{n-1}^{a(i)}\}$$

$a(i)$ is an ancestor.

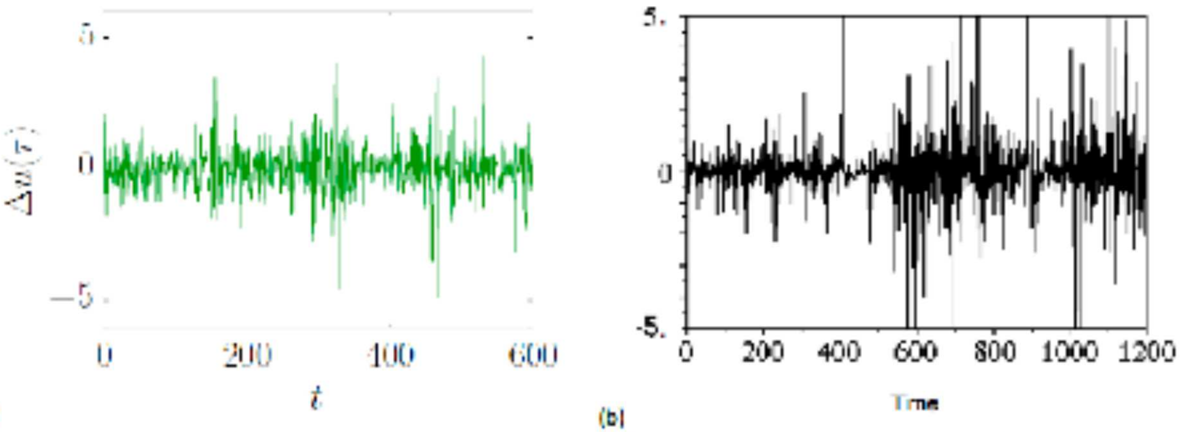
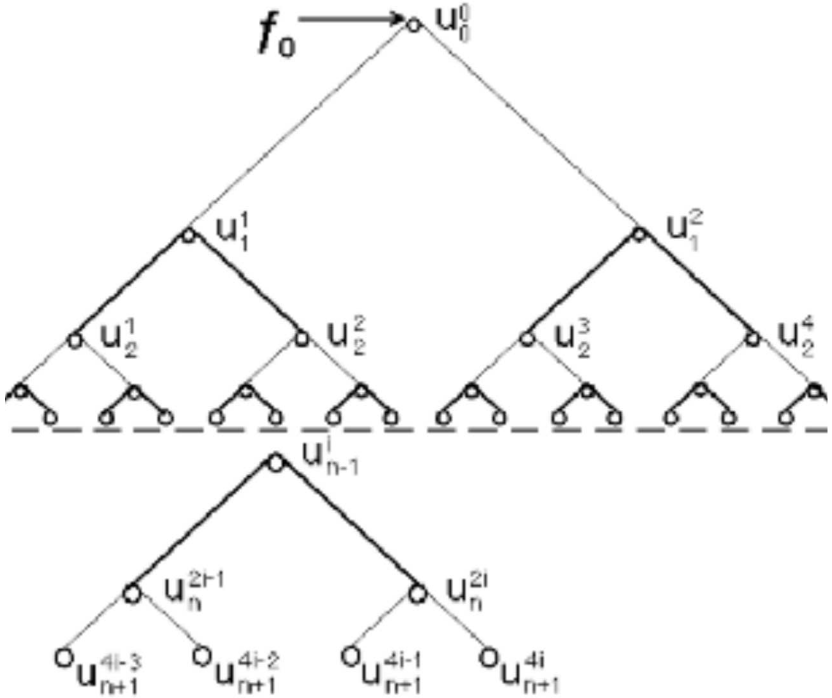


Figure 2: Comparison of fluctuations: (a) atmospheric turbulence at 100*m* (*Fitton*, 2013) and (b) SGC simulation for n = 6 (*Chigirinskaya and Schertzer*, 1996), both display somehow similar strong intermittency.



Local flux of energy:

$$\varepsilon_n^i=-\sum_{r=0}^{n}k_{n-r+1}\left[\left|\hat{u}_{n-r+1}^{2a^r(i)-1}\right|^2-\left|\hat{u}_{n-r+1}^{2a^r(i)}\right|^2\right]\mathrm{Im}(\hat{u}_n^{a^r(i)})+(-1)^{a^r(i)+1}k_{n-r}\left|\hat{u}_n^{a^{r+1}(i)}\right|^2\mathrm{Im}(\hat{u}_{n-1}^{a^{r+1}(i)})$$

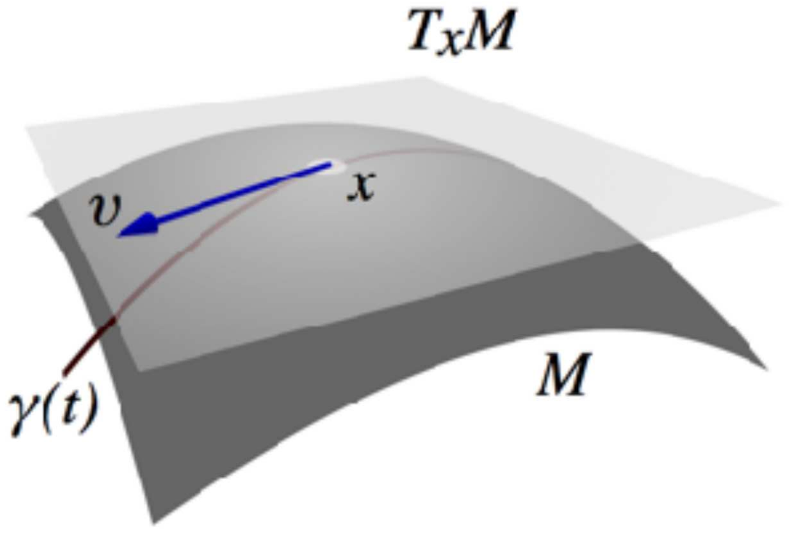
Mr Jourdain and Lie cascades

What is general and theoretically straightforward:

- $exp : Lie\ algebra \longmapsto Lie\ group$

scalar valued cascade: $R^d \dashrightarrow R^+$

- Lie group: smooth manifold
- Lie algebra: tangent space to the group at the identity
- therefore a vector space with a skew product that satisfy the Jacobi identity:



$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

exemple: commutator of matrices

$$[X,Y] = XY - YX \qquad [X,Y] = 0 \Rightarrow \exp(X+Y) = \exp(X)\exp(Y)$$

Mr Jourdain and Lie cascades

- Levi decomposition of any Lie algebra into its radical (*good guys*!) and a semi-simple subalgebra (*bad guys*!), e.g.:

$$l(2,R) = R\,1 \oplus_s sl(2,R)$$

What is trickier:

- large number of degrees of freedom (*dim*^{2})
- log divergence with the resolution
- universality:
  - Levy multivariates, unlike Gaussian mutivariates, are non parametric (*)
  - asymmetry of Levy noises to have convergent statistics, e.g.:

$$\forall n \in N, \forall X \geq 0: \ \exp(X) \geq X^n/n!$$

(S&L, 95, T&S 96)

(*) limitation of anamorphosis transform and/or geostatistics

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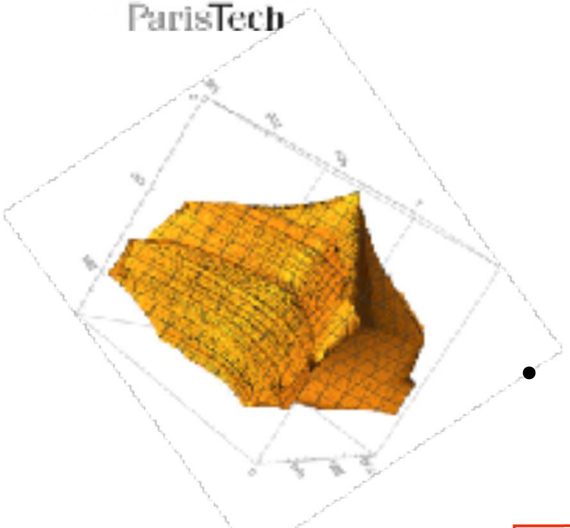


Mandelbrot set in hyperbolic geometry, S&T,2018

Clifford algebra

- An important family of Lie algebras:
  - dimension: 2^n
  - generalises real numbers *R* (*n*=0), complex numbers *C* (*n*=1), quaternions *H* (*n*=2) and other hyper-complex numbers, external algebras and more!
- $Cl_{p,q}$ generated by operators e^i that anti-commute and square to plus or minus the identity:

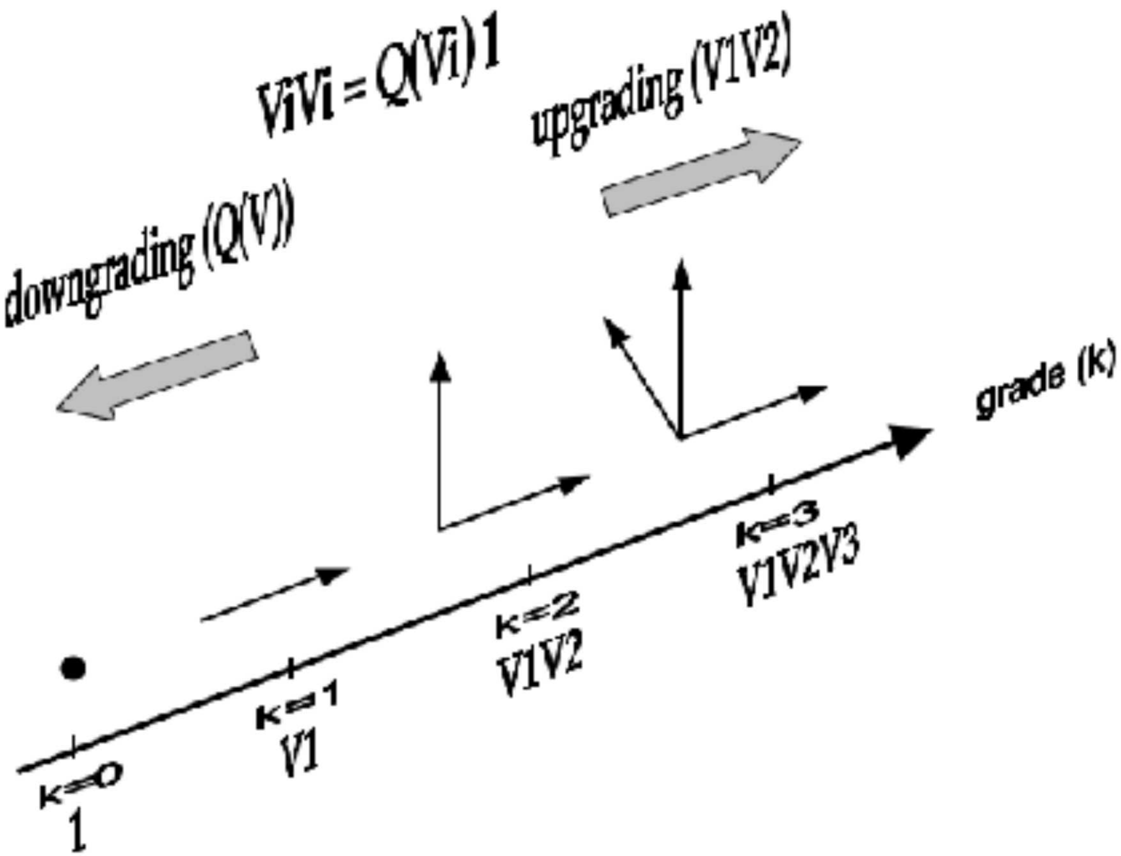
$$e^i e^j = -e^j e^i \ (i \neq j) \qquad (e^i)^2 = \pm 1$$

- hence a quadratic form Q, of signature (*p*,*q*, *p*+*q*=*n*):

$$v^2 = Q(v)1 \ \ Q(v) = v_1^2 + v_2^2 .. + v_p^2 - v_{p+1}^2 - v_{p+2}^2 .. - v_{p+q}^2$$

ex.: $R = Cl_{0,0}$; $C = Cl_{0,1}$; $H = Cl_{0,2}$
H'= $l(2, R) = Cl_{2,0} = Cl_{1,1}$ "pseudo-/split quaternions"

Clifford algebra



Clifford algebra are

- graded algebra (see figure)
- double algebra:
  - 2 multiplications
- super algebra (!):

$$Cl(V,Q) = Cl^0(V,Q) \oplus Cl^1(V,Q)$$

for real algebra:

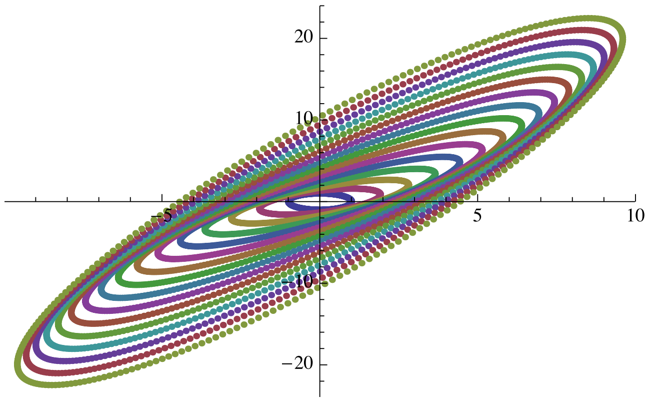
$$Cl^0_{p,q}(R) \cong Cl_{p,q-1}(R) \text{ for } q > 0$$

$$Cl^0_{p,q}(R) \cong Cl_{q,p-1}(R) \text{ for } p > 0$$

==> $R \subset C \subset H \subset O$ …

What can we learn from pseudo-quaternions?

2D linear Lie algebra *l*(2, R):



$$G = d\mathbf{1} + e\mathbf{I} + f\mathbf{J} + c\mathbf{K};$$

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$2I = [J, K], \quad 2J = [I, K], \quad 2K = [J, I]$$

anti-commutators:

$$\{I, J\} = \{J, K\} = \{K, I\} = 0$$

$$-1 = I^2 = -J^2 = -K^2 = IJK$$

Fractionnaly Integrated Flux model (FIF, vector version)

FIF assumes that both the renomalized propagator G_R and force f_R are known:

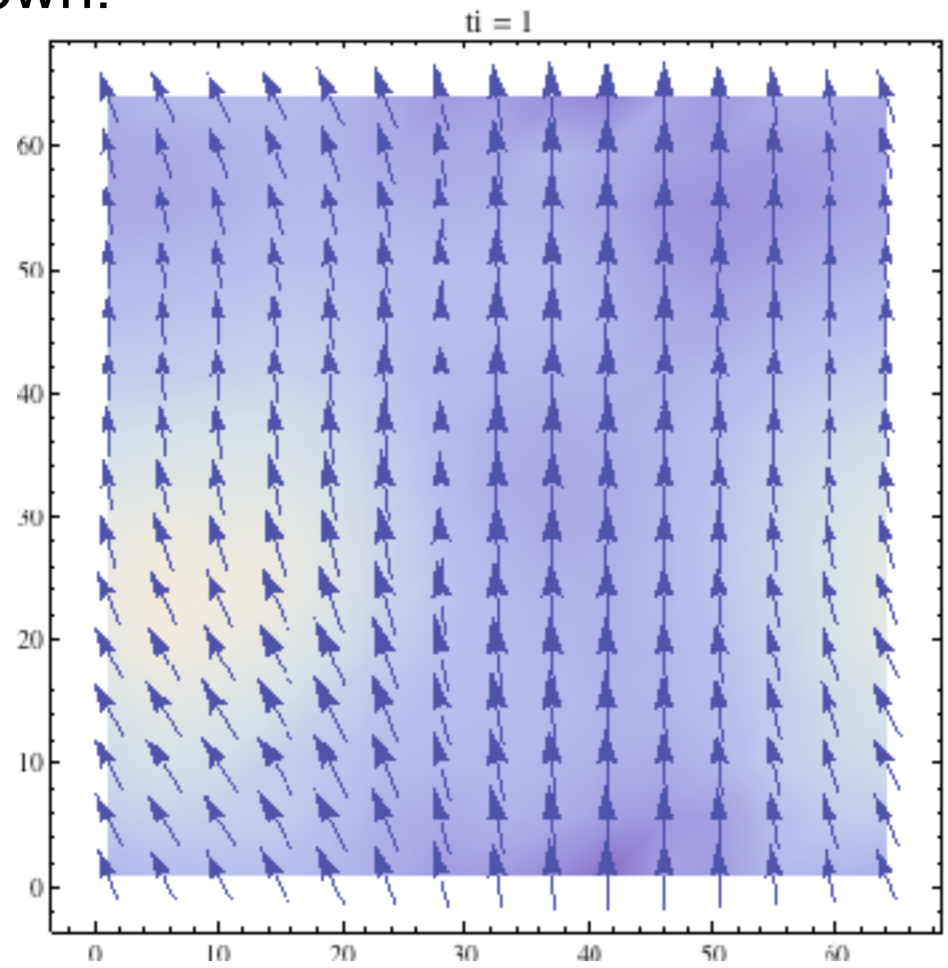
$$G_R^{-1} * u = f_R$$

where: $f_R = \varepsilon^a$

G_R^{-1} is a fractionnal differential operator

ε results from a continuous, vector, multiplicative cascade (Lie cascade)

Complex FIF simulation of a 2D cut of wind and its vorticity (color)



Fractionnaly Integrated Flux model (FIF, vector version)

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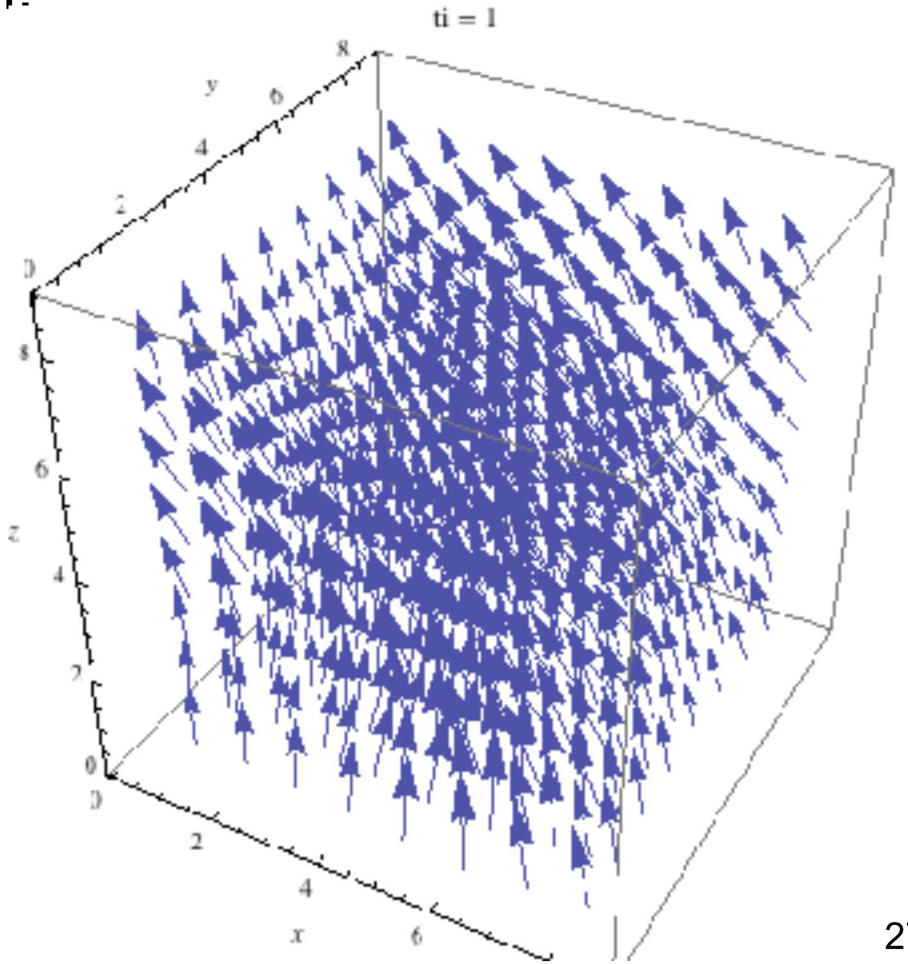
$$G_R^{-1} * u = f_R$$

where: $f_R = \varepsilon^a$

G_R^{-1} is a fractionnal differential operator

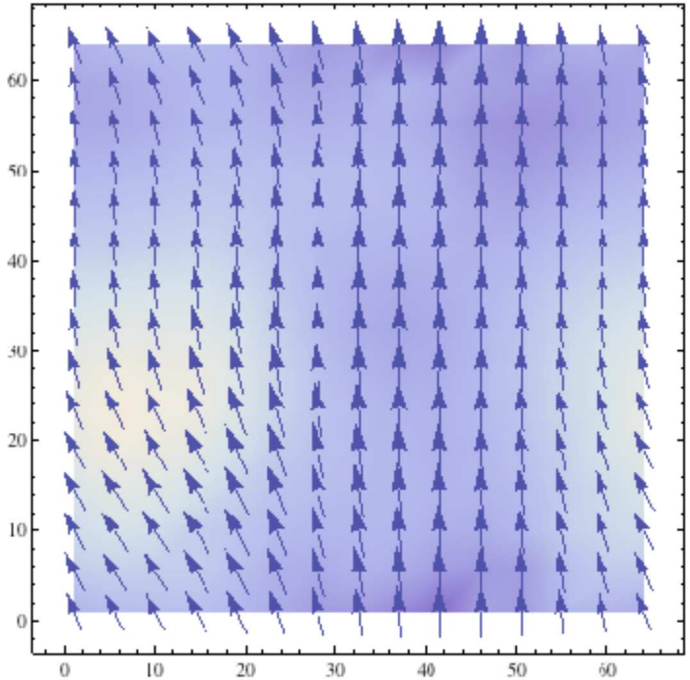
$\mathcal{E}$ results from a continuous, vector, multiplicative cascade (Lie cascade)

3D FIF wind simulation based on quaternions



Surface layer complexity!

Art piece ‘Windswept’ (Ch. Sowers, 2012): 612 freely rotating wind direction indicators to help a large public to understand the complexity of environment near the Earth surface

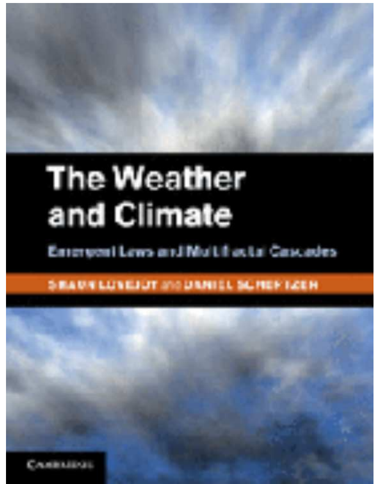


Multifractal FIF simulation (S et al., 2013) of a 2D+1 cut of wind and its vorticity (color). This stochastic model has only a few parameters that are physically meaningful.

Both movies illustrate the challenge of the near surface wind that plays a key role in the heterogeneity of the precipitations... and wind energy!

Conclusions

S&T, Earth& Space, 2020
Chaos 2015, S&al. ACP, 2012,
S&L, IJBC, 2011,
Fitton&al., JMI 2013



- Intermittency: a key issue in geophysics and a major breakthrough with multifractals in the 1980's:
  - infinite hierarchy of fractal supports of the field singularities
  - anisotropy: 2-D and 3-D turbulence are not the main options (2+*Hz*)-D atmospheric turbulence ($0 \le Hz \le 1$), with a theoretical $Hz=5/9$ resulting from strongly nonlocal interactions: classical;
- Not limited to scalar fields
  - Lie cascades: exponentiation from a stochastic Lie algebra to its Lie group of transformations
  - Clifford algebra $Cl_{p,q}$ mentioned at once..
  - now used for vector fields
  - => physically meaningful and convenient to understand, analyse and simulate intermittent vector fields, more generally multidimensional systems.

=> from field physics to singularity physics